MATH 148 **Assignment 6** Due: Friday, March 11

Please get started on this assignment early. If you wait until the later next week, then you might get frustrated with some of the problems.

1. Here is a neat problem in applied mathematics that illustrates the usefulness of integration. In various controlled circumstances a biological population $p(t)$ at time t grows at a rate that is jointly proportional to the amount of population present as well as the amount of room left for the population to grow. Say the maximum population that a given environment can support is M. Then it makes some sense to argue that if $p(t)$ is big while the room to grow $M - p(t)$ is still a good size, then the rate of growth $p'(t)$ will be substantial. This is because there are lots of individuals in $p(t)$ to "make babies", and there is lots of space in $M - p(t)$ for the babies to thrive. On the other hand if $p(t)$ is very small, then there is not enough population to reproduce rapidly. At the other end of things if $M - p(t)$ is tiny, then there is hardly any room for the population to grow and again $p'(t)$ will be small. To put it mathematically, there are some situations where the population $p(t)$ of a species in an environment satisfies the differential equation

$$p'(t) = p(t)(M - p(t)) \text{ with the conditions } 0 < p(t) < M. \qquad (1)$$

   (a) If the starting population at time 0 is $p(0) = 1$ (i.e take the starting population to be one unit of population), and the maximum possible population (a.k.a. the *carrying capacity*) is $M = 2$ find the population at any time t. In other words solve the differential equation (1) for $p(t)$ assuming $M = 2, p(0) = 1$ and $0 < p(t) < 2$.
   Hint. Show that (1) gives $\displaystyle\int_0^x \frac{p'(t)}{p(t)(2 - p(t))}\,dt = x$ for any time x. Substitute $u = p(t)$, change the endpoints appropriately, and keep chugging away until you find $p(x)$ for all real numbers x.

   (b) At what time is the population growing most rapidly? This can be figured out directly from the equation (1).

   (c) Sketch the graph of $p(x)$. There is no need to do a full derivative treatment, but show the horizontal asymptotes as time x tends to $\pm\infty$. By the way equation (1) tells you that $p'(t) > 0$ for all t (why?), so you know that your solution $p(t)$ had better be increasing, which is what populations tend to do until they start to hit their carrying capacity, or something else gets into the picture to wreck equation (1).

2. Test the following series for convergence.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^5}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$

(c) $\displaystyle\sum_{n=2}^{\infty} (\sqrt[n]{n}-1)^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi)\sin(1/n)$

(f) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{n}$

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

3. Suppose that $\dfrac{f(x)}{g(x)} = \dfrac{a_m x^m + \cdots + a_0}{b_k x^k + \cdots + b_0}$ is a rational function where $a_m > 0$ and $b_k > 0$. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{g(n)}$ converges if and only if $k \geq m+2$.

   Note to worriers. It is possible that $g(n)$ might vanish for a finite number of n, in which case it looks like you might accidentally be dividing by 0 for a while. Just ignore this little point and assume the series addition starts when the $g(n)$ stop being 0.

4. Apply a couple of condensations to decide if the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^2}$ converges.

5. Find a divergent series $\sum_{n=1}^{\infty} x_n$ for which the $x_n \to 0$ and the partial sums s_n stay bounded.
Hint. I might concentrate on finding a bounded sequence s_n that does not converge and for which $s_{n+1} - s_n \to 0$.

6. Use a limit comparison with $\sum 1/n^2$ to show that $\sum(1 - \cos(1/n))$ converges.

7. (a) Show that the series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ converges absolutely when $|x| < e$, and diverges when $|x| > e$.
Hint. Try the ratio test.
The next parts of this problem are aimed at deciding what happens when $x = \pm e$.

   (b) Use a simple diagram to explain the inequality

$$\int_1^n \ln(t)\, dt < \ln(2) + \ln(3) + \cdots + \ln(n) \text{ for } n = 2, 3, 4, \ldots.$$

   (c) Show that $n^n e^{1-n} < n!$ for $n = 2, 3, \ldots$.

   (d) Decide on the convergence of $\sum_{n=1}^{\infty} \frac{n! e^n}{n^n}$ and $\sum_{n=1}^{\infty} \frac{n!(-e)^n}{n^n}$.

8. If $\sum_{n=1}^{\infty} x_n$ converges absolutely, show that $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} \frac{x_n}{1 + x_n}$ also converge absolutely.

Hint. Explain why $|1 + x_n| > 1/2$ eventually, and the first one is easier.

9. (a) Show that $\sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n$ converges.

   (b) Using item(a) show that $\int_1^{\infty} \frac{e^x}{x^x}\, dx$ converges.

   (c) Does the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converge?
Hint. Item (b) ought to be good for something.

BONUSES

You can hand these in these optional problems by the end of March.

8. If f is continuous on $[0, \infty)$ and $f(x) \to a$ as $x \to \infty$, show that also $\frac{1}{x}\int_0^x f \to a$ as $x \to \infty$.

   This says that if the function is close to the asymptote value a in the long run, then in the long run its average value gets close to a too. This problem is not so hard.

   Does the converse of the above result hold?

9. If $s = \sum_{k=1}^{\infty} \frac{1}{k^3}$ and s_7 is the seventh partial sum, is $s - s_7 < \frac{1}{100}$?