MATH 148 **Assignment 7** Due: Friday, March 18

Please get started on this assignment early. If you wait until the later next week, then you might feel unpleasantly rushed.

1. For each series below find all x such that the series converges. In each case the ratio test should help sort out all but a couple of x. Then work on those couple of leftover cases.

   (a) $\sum_{n=1}^{\infty} nx^n$ (b) $\sum_{n=1}^{\infty} n^n x^n$

   (c) $\sum_{n=0}^{\infty} \frac{n}{4^n}(2x-1)^n$ (d) $\sum_{n=0}^{\infty} \frac{(n!)^p}{(pn)!}x^n$ where p is a fixed positive integer

2. (a) Show that

   $$-x^{2n} \leq \frac{1}{1+x^2} - \left(1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}(x^2)^{n-1}\right) \leq x^{2n}$$

   for each positive integer n.

   Hint. Add up the geometric series in the brackets.

   (b) Do an integration over $[0,1]$ of the above functions and compute the sum of the series

   $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots$$

3. The integral test says that if f is ≥ 0 and decreasing on the interval $[1,\infty)$, then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int_1^{\infty} f(t)\,dt$ converges.

   Find a continuous, non-negative function f on $[1,\infty)$ such that the integral converges but the series diverges, and find another f where the integral diverges but the series converges. For this problem examples based on nice pictures will make me happy enough, so there is no need for fancy formulas. Obviously such functions had better not be decreasing.

4. If x_n is a bounded sequence and c is a positive constant, show that

   $$\limsup cx_n = c \limsup x_n.$$

   This is routine stuff once you understand $\limsup$.

   Is it true that $\limsup(x_n + y_n) = \limsup x_n + \limsup y_n$?

5. Here's a bit of review on integration theory.

   A function f is *integrable* over some interval $[a, b]$ when there is *exactly* one number above all lower sums and below all upper sums for f. That one number is called the integral of f over $[a, b]$. If a function f is *continuous* over $[a, b]$ we know many things about f. Such as for example, f is uniformly continuous over $[a, b]$. This allows us to show that a continuous f is integrable over $[a, b]$. From this it follows that a continuous f is integrable over any subinterval $[a, x]$ of $[a, b]$. This allows us to define the integral function $g(x) = \int_a^x f$ for every x in $[a, b]$. Then the fundamental theorem of calculus part 2 tells us that for f continuous we have $g'(x) = f(x)$ for every x in $[a, b]$. Thus we come to the conclusion that every continuous function on an interval has an anti-derivative, and that its anti-derivative is given by its integral function.

   However, there are many integrable functions that are not continuous.

   (a) Find an integrable function f on $[-1, 1]$ whose integral function $g(x) = \int_{-1}^x f$ does not have a derivative at $x = 0$.

   (b) Find an integrable function f on $[-1, 1]$ whose integral function $g(x) = \int_{-1}^x f$ has a derivative at all x in $[-1, 1]$ but $g'(0) \neq f(0)$.

   There is no need to get too fancy with the examples requested above.

6. Take the following sequence of functions:

$$f_n(x) = x^n \text{ for } x \text{ in the interval } [0, 1].$$

   For each x in $[0, 1]$ find $\lim_{n\to\infty} f_n(x)$. If $f(x)$ is the limit you get for each x in $[0, 1]$, is the resulting function f continuous on $[0, 1]$. Explain briefly.

The rest of this assignment is about **power series**. There are some facts discussed below you will need to know. These will be proven in class soon. In the meantime it's a good idea to start working with power series, before we run out of time.

For each number x and fixed coefficients $a_0, a_1, a_2, \ldots$ the series

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{k=0}^{\infty} a_kx^k$$

is called a *power series* in x. We are bound to seek those x for which such a series converges. If a series converges for some x, we can let the sum of the series be called $f(x)$. Thus power series have the potential to make functions. The first fundamental result to know (and which we will prove in class) is that every power series MUST do one of three things:

- converge just for $x = 0$ (this is the useless case)
- converge absolutely for all x
- converge absolutely when $|x| <$ some positive number R and diverge when $|x| > R$.

In the last case, the positive number R is called the *radius of convergence* of the power series. In the first case, we say that the radius is 0, and in the second case, the radius is said to be ∞.

For example, the ratio test applied to the series $\sum_{k=0}^{\infty} k!x^k$ shows that this series converges just for $x = 0$, so the radius is 0. The series $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ has radius ∞ because we can use the ratio test to prove it converges for all x. The function that this series converges to is usually called e^x. As a third example, we know that $\sum_{k=0}^{\infty} x^k$ converges if and only if $|x| < 1$. So for this last geometric series the radius is 1. This geometric series converges to the function $\frac{1}{1-x}$.

A power series with a strictly positive radius R converges when $|x| < R$ to a value $f(x)$. In this way power series are said to *represent* functions $f(x)$ on the interval $(-R, R)$. This idea of a series representing a function on an interval $(-R, R)$ is very important.

You might also notice that question # 1 was also about power series.

7. Using the ratio or the root test find the radius of convergence of the following series

$$\text{(a)} \sum_{n=2}^{\infty} \frac{1}{(\ln n)^2} x^n \qquad \text{(b)} \sum_{n=2}^{\infty} \frac{1}{(\ln n)^n} x^n \qquad \text{(b)} \sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln n}} x^n$$

8. Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R where $0 < R < \infty$, and let k be a fixed positive integer. What is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^{kn}$?

   Note. In this power series, we take to be 0 the coefficients of those powers of x that do not appear.

9. Using your knowledge of geometric series, and in part (c) partial fractions, find power series to represent the following functions. In each case specify the radius of convergence of your series.

   (a) $\dfrac{1}{2+5x}$ (b) $\dfrac{1}{3x^2+1}$ (c) $\dfrac{1}{2x^2-3x+1}$

   Hint. One fact I always remember is that

   $$1 + t + t^2 + \cdots + t^n + \cdots = \frac{1}{1-t} \text{ when } |t| < 1.$$

   So try to get expressions that involve "$\frac{1}{1-t}$"and plug into the above fact.

   Geometric series are truly important. They give us access to numerous other power series.

BONUS

You can hand these in these optional problems by the end of March.

10. This question popped into my head when I was writing up question #5 above. At this moment I don't know if it is easy, hard, known, or unknown. I would guess it is rather hard but known.

    Suppose F is a differentiable function on $[0, 1]$ and that $f(x) = F'(x)$ for all x in $[0, 1]$ and that f is bounded on $[0, 1]$. Must f be integrable on $[0, 1]$? When you look at the proof of FTC part 1, you will see that we assumed f was integrable. Can this assumption be disposed of, or is there a bounded f that is the derivative of some F and yet f is not integrable?

    These kinds of questions involving the pathology of functions are of a kind that you either love them a lot or really, really hate them.