MATH 148 **Assignment 8** Due: Friday, March 25

This assignment is longer and more lively than normal. It touches on a lot of core material. To avoid frustrations, please start working on it right away.

A couple of special results about power series are the *integration and differentiation theorems*. Even though the proofs are not easy, the results are easy to state. First digest what they say below.

Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots \text{ with radius } R. \tag{1}$$

The *integrated series* of f is defined to be:

$$a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_n}{n+1} x^{n+1} + \ldots \tag{2}$$

In other words just integrate each term in the expansion (1) of f. The *integration theorem* for series says that the integrated series (2) has the same radius R as the original series in (1), and that the integrated series in (2) converges to the integral function $\int_0^x f(t)\,dt$. In other words, the integrated series represents the integral function and the radius of convergence does not change.

The *derived series* of f is defined to be:

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^4 + \cdots + n a_n x^{n-1} + \ldots \tag{3}$$

In other words, just differentiate each term in the expansion (1) of f. The *differentiation theorem* says that the derived series given in (3) has the same radius R as the original series (1) for f, and converges to the derivative $f'(x)$.

Feel free to use these theorems as needed in the exercises that follow.

1. Use the root test to find the radius of the series

$$1 + x + 2x^2 + \frac{1}{3}x^3 + 4x^4 + \frac{1}{5}x^5 + 6x^6 + \frac{1}{7}x^7 + \ldots$$

   Find an explicit formula for the function represented by this series.

   Hint. Find the sums of the series involving even and odd powers of x separately. To do that take the known expansion for the geometric series, and then use the differentiation and integration theorems for series, along with a bit of algebraic trickery.

2. Let us recall the *binomial theorem*. It said that if $n = 0, 1, 2, 3, \ldots$, then

$$(1+x)^n = 1 + nx + \binom{n}{2} x^2 + \cdots + \binom{n}{r} x^r + \cdots + x^n,$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \quad \text{for } r = 0, 1, 2, \ldots, n.$$

Here we explore what the binomial theorem says when $n \neq 0, 1, 2, 3, \ldots$.

We take **any** fixed real number $t \in \mathbb{R}$ but $t \neq 0, 1, 2, 3, \ldots$.

If $r = 1, 2, 3, \ldots$ the binomial coefficient $\binom{t}{r}$ is defined to be the number

$$\binom{t}{r} = \frac{t(t-1)(t-2)\cdots(t-r+1)}{r!}. \text{ We also put } \binom{t}{0} = 1.$$

Notice now that the numerators need not be integers, and that there is no bound on big the integer r can be.

(a) Just to warm up, calculate and simplify $\binom{-2}{6}$, $\binom{1/2}{5}$ and $\binom{-1/3}{4}$.

(b) Verify by grinding out the definition of binomial coefficients that

$$(r+1)\binom{t}{r+1} + r\binom{t}{r} = t\binom{t}{r} \quad \text{for all } r = 0, 1, 2, \ldots.$$

(c) For each t use the ratio test to show that the radius of convergence of the series

$$1 + tx + \binom{t}{2} x^2 + \binom{t}{3} x^3 + \cdots + \binom{t}{r} x^r + \ldots$$

is 1. Where did you need the fact $t \neq 0, 1, 2, \ldots$?

(d) If $f(x) = \sum_{n=0}^{\infty} \binom{t}{r} x^r$ where $x \in (-1, 1)$, we are interested in figuring out the function f in more familiar terms.

Use the differentiation theorem along with part (b) up above to show that

$$(1+x)f'(x) = tf(x) \quad \text{for all } x \text{ in } (-1, 1)$$

(e) Prove that $\left(\frac{f(x)}{(1+x)^t}\right)' = 0$, and use this information to deduce that

$$f(x) = (1+x)^t \text{ for } -1 < x < 1.$$

Thus you have derived and verified what is known as Newton's binomial expansion for exponents t that are not $0, 1, 2, \ldots$.

This is a really neat result!

3. If the radius of the power series $\sum_{n=0}^{\infty} a_n x^n$ is R and $0 < R < \infty$, what is the radius of $\sum_{n=0}^{\infty} a_n x^{n^2} = a_0 + a_1 x^1 + a_2 x^4 + a_3 x^9 + a_4 x^{16} + \ldots$?

4. Suppose

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \ldots \text{ with radius } R.$$

The differentiation theorem as discussed above for a function f can be reapplied to f' to get a formula for f'' and then for f''' and so on.

(a) Use this to prove that

$$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \frac{f''(0)}{2},\ a_3 = \frac{f'''(0)}{6}, \ldots, a_n = \frac{f^{(n)}(0)}{n!}, \ldots$$

Of course you now seee that the coefficients of a power series that represents a function must be the Taylor coefficients of the function.

(b) Explain why two different power series cannot represent the same function on an interval $(-R, R)$.

5. We just saw above that if f is a function given by a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on some interval $(-R, R)$ where $0 < R \leq \infty$, then f has derivatives of all orders for all x in $(-R, R)$, and that the coefficients of the series representing f have to be the Taylor coefficients of f.

A decent question concerns the converse of this. Suppose f has derivatives of all orders on some interval $(-R, R)$, is there a power series representation of f on $(-R, R)$? Regrettably NOT. For instance take the function $f(x) =$

$\dfrac{1}{1+x^2}$ on $(-\infty, \infty)$, which has derivatives of all orders on $(-\infty, \infty)$. It has the power series representation

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \ldots,$$

but this only holds for x in $(-1, 1)$ instead of $(-\infty, \infty)$.
The next example will show you that things can get much worse than this.

Let $f(x) = \begin{cases} e^{-1/x^2} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$.

It is easy to see that f is continuous at 0. A look at the graph of f using MAPLE can be quite informative and is **strongly recommended**. Answer the following questions regarding this f.

(a) First show by induction on $n = 0, 1, 2, \ldots$ that if $x \neq 0$, then $f^{(n)}(x) = p\left(\frac{1}{x}\right) e^{-1/x^2}$ where $p\left(\frac{1}{x}\right)$ is a polynomial in $\frac{1}{x}$.

(b) Show that f has derivatives of all orders at 0 and that $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$.

Hint. Use induction on n along with the facts $e^{-x}/x^k \to 0$ as $x \to \infty$ (regardless of k), and thus $\frac{1}{x^k} e^{-1/x^2} \to 0$ as $x \to 0$, and thus, for any polynomial in $\frac{1}{x}$, $p\left(\frac{1}{x}\right) e^{-1/x^2} \to 0$ as $x \to 0$.

(c) Suppose that $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ for x in some open interval $(-R, R)$.

Obtain a contradiction by showing all $a_n = 0$.

Hint. Use problem 4.

6. You may have wondered why odd functions are called "odd" and even functions are called "even". Here is a possible explanation.

   Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ for x in an interval $(-R, R)$, and that f is an even function on the interval $(-R, R)$.

   Prove that $a_1 = a_3 = a_5 = a_7 = \cdots = 0$.

   Thus only the even coefficients appear in the power series expansion of f.

   Likewise prove that if f is odd, then $a_0 = a_2 = a_4 = a_6 = \cdots = 0$.

   Hint. Not so hard using problem 4 above.

7. Knowing the standard geometric power series representation for $\frac{1}{1-x}$ on $(-1,1)$, you certainly can come up with power series representations for $\frac{1}{1+x}, \frac{1}{1+x^2}$ and stuff like that.

   (a) Use the integration theorem discussed in the previous assignment to find a power series representation for $f(x) = \ln(1+x)$ on $(-1,1)$.

   (b) Use part (a) above, but not a calculator, to find a fraction that estimates $\ln(3/2)$ with error at most $1/64$.

   (c) Find a power series representation of $g(x) = \arctan\left(\frac{x^2}{2}\right)$, and give its radius of convergence.

8. Start with the usual formula for the sum of a geometric series. Apply the differentiation theorem to it two times. You now have a series that converges to a known analytic function on $(-1,1)$. Multiply this series by x^2, and thereby represent another analytic function on $(-1,1)$. Use this information to find the value of $\sum_{n=0}^{\infty} \frac{n^2}{3^n}$.

9. Use the power series representation of e^x, in conjunction with the error estimate in the alternating series test, to estimate $\int_0^{1/2} e^{-x^3}\,dx$ with error at most $1/10^9$.

10. By the binomial expansion of the previous assignment you know there are coefficients a_n such that

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = a_0 + a_1 x + a_2 x^2 + \ldots \quad \text{for } x \in (-1,1).$$

   (a) Find a_0, a_1 and a_2.

   (b) By a simple substitution show that $\frac{1}{\sqrt{1-x^2}}$ is analytic on $(-1,1)$ and find its power series expansion as far the x^4 term.

   (c) Show that $\arcsin(x)$ is analytic on $(-1,1)$ and find its power series expansion up to the x^5 term. Then write the power series expansion of $\arcsin(x^2)$ as far as the x^{10} term.

(d) It is well known that $\sin(x)$ for every x in $\mathbb{R}$:

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots.$$

Maybe you did this in MATH 147.

Use this information along with the parts done before to find

$$\lim_{x\to 0} \frac{\arcsin(x^2) - \sin(x^2)}{x^6}.$$

11. For each sequence of functions below on the interval specified decide if the sequence converges point-wise, determine the limit function, and then decide if the convergence is uniform.

(a) $f_n(x) = nx^n(1-x)$ on $[0,1]$

(b) $f_n(x) = \dfrac{nx}{1+n+x}$ on $[0,\infty)$

(c) $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ on $\mathbb{R}$

BONUS

You can hand in this optional problem by April 1.

11. Suppose that on the closed interval $[0,1]$ you have a sequence of *non-negative, continuous* functions f_n and that for every x in $[0,1]$ the sequence $f_n(x)$ decreases and converges to 0. Show that $f_n \to 0$ uniformly on $[0,1]$.

Suggestions. Explain why $\|f_1\| \geq \|f_2\| \geq \dots \geq \|f_n\| \geq \dots$. If $\|f_n\| \not\to 0$, show there is a sequence x_n in $[0,1]$ such that $f_n(x_n)$ is bounded away from 0. Apply Bolzano-Weierstrass.