MATH 148 **Assignment 9** Due: Friday, April 1

1. Find $\lim_{x \to 0} \dfrac{\arctan(2x^2) - 2(\sin x)^2}{x \ln(1 + x^3)}$.

2. A function given by a power series $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$ satisfies the differential equation

$$f''(x) + f'(x) + f(x) = 1 \text{ and } f(0) = 0, f'(0) = 1.$$

Find the first five terms of the power series expansion of f.

3. Use power series to show that $f(x) = \dfrac{\cos(x) - \sqrt{1 + x^2}}{x^2}$ can be defined at 0 in such a way that the resulting function has derivatives of all orders at 0. Then calculate $f^{(4)}(0)$.

4. Using partial fractions you can calculate a power series expansion for $f(x) = \dfrac{x}{1 - x - x^2}$. With a bit of patience it is doable, even though the calculations get messy as they involve the golden ratio $\dfrac{1 + \sqrt{5}}{2}$. Suppose you did the work and got $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with some positive radius R. Prove that

$$a_0 = a, a_1 = 1, a_2 = 1, \text{ and then } a_n = a_{n-1} + a_{n-2} \text{ for } n = 3, 4, 5, \dots.$$

Thus you learn that the coefficients of the power series that represents f are the world famous Fibonacci numbers. If you *actually* calculate the coefficients explicitly, you will get an explicit formula for the n'th Fibonacci number in terms of the golden ratio. This is neat stuff.

5. Show there is a sequence f_n of integrable functions on the interval $[0, 1]$ that converges pointwise to the non-integrable function which is 1 at every rational and 0 at every irrational.

6. Show that $\sum_{n=1}^{\infty} \dfrac{1}{1 + n^2 x}$ converges uniformly on every interval $[a, \infty)$ where $a > 0$, but not uniformly on $(0, \infty)$.

7. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$ converges uniformly on $[0, \infty)$.

   Note. When you try to apply the M-test you will get frustrated looking for suitable M_n's. Try using something about the error terms for alternating series.

8. Prove that the series $\sum_{n=0}^{\infty} \frac{nx}{1+n^4x^2}$ does not converge uniformly on $[0, \infty)$.
   This is can get sticky.