MATH 245 Linear Algebra 2, Assignment 1 Due Fri May 27

1: (a) Let A be a set. For each $\alpha \in A$, let $p_\alpha \in \mathbf{R}^n$, let U_α be a vector space in $\mathbf{R}^n$, and let $P = \bigcap_{\alpha \in A} (p_\alpha + U_\alpha)$. Show that if P is not empty then it is an affine space in $\mathbf{R}^n$.

(b) Let $p = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \\ 2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 1 & -1 & -2 \\ 1 & -3 & 0 & -1 & 3 \end{pmatrix}$ and $q = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, and let $P = p + \mathrm{Col}A$ and $Q = \left\{ x \in \mathbf{R}^5 \middle| Bx = q \right\}$. Show that $P \cap Q$ is not empty and find its dimension.

2: For two affine spaces P and Q in $\mathbf{R}^n$, the **distance** between P and Q is defined to be

$$\mathrm{dist}(P,Q) = \min \left\{ \mathrm{dist}(x,y) \middle| x \in P, y \in Q \right\}.$$

(a) Let p and q be points in $\mathbf{R}^n$, let U and V be subspaces of $\mathbf{R}^n$, and let $P = p + U$ and $Q = q + V$. Show that

$$\mathrm{dist}(P,Q) = \left| \mathrm{Proj}_{(U+V)^\perp}(p-q) \right|.$$

(b) Let $p = (1,2,4,3)^t, u_1 = (1,2,0,1)^t, u_2 = (3,5,1,2)^t, q = (4,3,1,2)^t, v_1 = (2,3,2,-1)^t, v_2 = (1,3,1,-2)^t$. Find the distance between the plane $x = p + t_1 u_1 + t_2 u_2$ and the plane $x = q + s_1 v_1 + s_2 v_2$.

3: For two non-trivial vector spaces U and V in $\mathbf{R}^n$, we define the **angle** between U and V, which we write as $\mathrm{angle}(U,V)$, as follows. If $U \subset V$ or $V \subset U$ then $\mathrm{angle}(U,V) = 0$, otherwise if $U \cap V = \{0\}$ then

$$\mathrm{angle}(U,V) = \min \left\{ \theta(u,v) \middle| 0 \neq u \in U, 0 \neq v \in V \right\},$$

and if $U \cap V = W \neq \{0\}$ then $\mathrm{angle}(U,V) = \mathrm{angle}(U \cap W^\perp, V \cap W^\perp)$. We define the angle between two affine spaces in $\mathbf{R}^n$ to be the angle between their associated vector spaces.

(a) Let $0 \neq u \in \mathbf{R}^n$, let $U = \mathrm{Span}\{u\}$, and let V be a non-trivial vector space in $\mathbf{R}^n$. Show that

$$\mathrm{angle}(U,V) = \cos^{-1} \left| \mathrm{Proj}_V \left(\frac{u}{|u|} \right) \right|.$$

(b) Let $u_1 = (1,-2,1,-3)^t$, $u_2 = (3,2,1,-1)^t$, $v_1 = (1,-1,0,1)^t$, $v_2 = (1,-3,2,-1)^t$ and $v_3 = (2,-1,1,-1)^t$. Find the angle between $U = \mathrm{Span}\{u_1,u_2\}$ and $V = \mathrm{Span}\{v_1,v_2,v_3\}$.

4: Let $a_0, a_1, \cdots, a_l$ be points in $\mathbf{R}^n$. Show that $\left[a_0, a_1, \cdots, a_l\right] = \left\{ \sum_{i=0}^{l} s_i a_i \middle| 0 \leq s_i \in \mathbf{R}, \sum_{i=0}^{l} s_i = 1 \right\}$.

5: Let $S = \left[a_0, a_1, \cdots, a_l\right]$ be an l-simplex in $\mathbf{R}^n$. for each $0 \leq j < k \leq n$, the **altitudinal hyperplane** $B_{j,k}$ is the $(l-1)$-dimensional affine space in $\langle a_0, a_1, \cdots, a_l \rangle$ which is perpendicular to the edge $[a_j, a_k]$ and which passes through the centroid of the $(l-2)$-simplex $\left[a_0, a_1, \cdots, \check{a}_j, \cdots, \check{a}_k, \cdots, a_l\right]$, (where the check mark above the points a_j and a_k indicates that these points are excluded). Show that the altitudinal hyperplanes have a unique point of intersection. This point is denoted by h and is called the **orthocenter** of the l-simplex S.

(b) Let $S = \left[a_0, a_1, \cdots, a_l\right]$ be an l-simplex in $\mathbf{R}^n$. Let o, g and h be the circumcenter, the centroid, and the orthocenter of S. Show that g lies $\frac{l-1}{l+1}$ of the way along the line segment from o to h.