MATH 245 Linear Algebra 2, Assignment 3 Due Mon July 4

1: Let $u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ 1 \\ 7 \\ 3 \end{pmatrix}$. Let $\mathcal{U} = \{u_1, u_2, u_3\}$ and let $U = \text{Span}\,\mathcal{U}$. Find $\text{Proj}_U(x)$ in the following three ways.

(a) Let $A = \big(u_1, u_2, u_3\big) \in M_{4\times 3}$ then use the formula $\text{Proj}_U(x) = Ay$ where y is the solution to $A^t A\, y = A^t x$.

(b) Apply the Gram-Schmidt Procedure to the basis $\mathcal{U}$ to obtain an orthogonal basis $\mathcal{V} = \{v_1, v_2, v_3\}$ for U, then use the formula $\text{Proj}_U(x) = \sum_{i=1}^{3} \frac{x \cdot v_i}{|v_i|^2}\, v_i$.

(c) Find $w = X(u_1, u_2, u_3)$ so that $\{w\}$ is a basis for $U^\perp$, then calculate $\text{Proj}_U(x) = x - \text{Proj}_w(x)$.

2: Consider the vector space $P_2 = P_2(\mathbf{R})$ as a subspace of the vector space

$$\mathcal{C}\big((0,1)\big) = \mathcal{C}\big((0,1), \mathbf{R}\big) = \left\{ f : (0,1) \to \mathbf{R} \,\middle|\, f \text{ is continuous, and } \int_0^1 f(x)\,dx \text{ converges.} \right\}$$

with the inner product given by $\langle f, g\rangle = \int_0^1 fg$.

(a) Let $p_0 = 1$, $p_1 = x$ and $p_2 = x^2$. Apply the Gram-Schmidt Procedure to the basis $\{p_0, p_1, p_2\}$ to obtain an orthogonal basis $\{q_0, q_1, q_2\}$ for P_2.

(b) Find the quadratic $f \in P_2$ which minimizes $\int_0^1 \big(f(x) - x^{-1/2}\big)^2\, dx$.

(c) Given that $f \in \mathcal{C}\big((0,1)\big)$ with $\int_0^1 f(x)\,dx = 3$, $\int_0^1 x f(x)\,dx = 2$ and $\int_0^1 x^2 f(x)\,dx = 1$, find the minimum possible value for $\int_0^1 f(x)^2\,dx$.

3: Let U and V be inner product spaces over $\mathbf{R}$. An **isometry** from U to V is a surjective map $F : U \to V$ which preserves distance, so that for all $x, y \in U$ we have $\big|F(x) - F(y)\big| = |x - y|$. An inner product space **isomorphism** from U to V is a bijective linear map $G : U \to V$ which preserves inner product, so that for all $x, y \in U$ we have $\big\langle G(x), G(y)\big\rangle = \langle x, y\rangle$. Show that, in the case that U and V are finite dimensional, every isometry $F : U \to V$ is of the form $F(x) = G(x) + b$ for some inner product space isomorphism G and some $b \in V$.

4: Identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ using the bijection $\phi : \mathbf{C}^n \to \mathbf{R}^{2n}$ given by

$$\phi\big(x_1 + i\,y_1, \cdots, x_n + i\,y_n\big)^t = \big(x_1, y_1, \cdots, x_n, y_n\big)^t .$$

(a) Determine whether, for all vectors $u, v \in \mathbf{C}^n$, u is orthogonal to v in $\mathbf{C}^n$ if and only if $\phi(u)$ is orthogonal to $\phi(v)$ in $\mathbf{R}^{2n}$.

(b) Determine whether, for all complex subspaces $U, V \subset \mathbf{C}^n$, U is orthogonal to V in $\mathbf{C}^n$ if and only if $\phi(U)$ is orthogonal to $\phi(V)$ in $\mathbf{R}^{2n}$.

5: Identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ using the map ϕ from question 4. Given two 1-dimensional complex subspaces $U, V \subset \mathbf{C}^n$, we define the **angle** between U and V to be

$$\text{angle}(U, V) = \cos^{-1} \frac{|\langle u, v\rangle|}{|u||v|} \quad , \quad \text{where } 0 \neq u \in U \ , \ 0 \neq v \in V\,.$$

(a) Explain why this definition is well-defined.

(b) Determine whether, for all 1-dimensional complex subspaces $U, V \subset \mathbf{C}^n$, the angle between U and V in $\mathbf{C}^n$ is equal to the angle between $\phi(U)$ and $\phi(V)$ in $\mathbf{R}^{2n}$.