MATH 245 Linear Algebra 2, Assignment 4 Due Mon July 18

1: For $0 \neq u \in \mathbf{R}^3$ and $\theta \in \mathbf{R}$, let $R_{u,\theta} : \mathbf{R}^3 \to \mathbf{R}^3$ denote the rotation about the vector u by the angle θ (where the direction of rotation is determined by the right-hand rule: the right thumb points in the direction of u and the fingers curl in the direction of rotation).

(a) Let $u = \left(1, 1, -1\right)^t$ and let $\theta = \frac{\pi}{3}$. Find $A = \left[R_{u,\theta}\right]$.

(b) Let $B = \begin{pmatrix} 2 & 3 & -6 \\ -3 & 6 & 2 \\ 6 & 2 & 3 \end{pmatrix}$. Find $c > 0$, $0 \neq u \in \mathbf{R}^3$ and $0 \leq \theta \leq \pi$ such that $B = \left[c\, R_{u,\theta}\right]$.

2: (a) Let $A = \begin{pmatrix} 0 & & & \\ \vdots & & I & \\ 0 & & & \\ a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix} \in M_{n\times n}(\mathbf{C})$. Find $f_A(t)$ and find a basis for each eigenspace E_λ.

(b) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{pmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.

(c) Let $x_0 = 2$, $x_1 = 2$ and $x_2 = 1$, and for $n \geq 0$ let $x_{n+3} = 6x_n + 5x_{n+1} - 2x_{n+2}$. Use part (b) to find x_n.

3: Let $A \in M_{n\times n}(\mathbf{R})$. Suppose that A is diagonalizable over $\mathbf{C}$, so there exists a diagonal matrix $D \in M_{n\times n}(\mathbf{C})$ and an invertible matrix $Q \in M_{n\times n}(\mathbf{C})$ such that $Q^{-1}AQ = D$. Show that there exists an invertible matrix $P \in M_{n\times n}(\mathbf{R})$ such that $P^{-1}AP$ is in the block-diagonal form

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_k & & & & & & \\ & & & a_1 & b_1 & & & & \\ & & & -b_1 & a_1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & a_l & b_l \\ & & & & & & -b_l & a_l \end{pmatrix}$$

where each 1×1 block corresponds to a real eigenvalue λ_j of A, and each 2×2 block corresponds to a pair of conjugate complex eigenvalues $a_j \pm i\, b_j$.

4: (a) Let U and V be inner product spaces over $\mathbf{C}$. Let $L : U \to V$ be a linear map, and suppose that the adjoint $L^* : V \to U$ exists. Show that $\text{Null}(L^*L) = \text{Null}(L) = \text{Range}(L^*)^\perp$.

(b) Let U be an inner product space over $\mathbf{C}$. Let $L : U \to U$ be linear and suppose that L^* exists. Show that $L = L^* \iff \langle L(x), x\rangle \in \mathbf{R}$ for all $x \in U$.

5: Let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Let V be the inner product space over $\mathbf{F}$ consisting of all sequences $a = (a_1, a_2, a_3, \cdots)$ with each $a_k \in \mathbf{F}$ such that only finitely many of the terms a_k are non-zero, with the inner product given by $\langle a, b\rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$. Let $U = \left\{ a = (a_1, a_2, \cdots) \in V \,\middle|\, \sum_{k=1}^{\infty} a_k = 0 \right\}$. The standard basis for V is the basis $\mathcal{S} = \{e_1, e_2, e_3, \cdots\}$ where $e_n = (e_{n,1}, e_{n,2}, e_{n,3}, \cdots)$ with $e_{n,k} = \delta_{n,k}$.

(a) Show that $U^\perp = \{0\}$.

(b) Show that $\dim(U^0) = 1$.

(c) Let $\mathcal{F} = \left\{f_1, f_2, f_3, \cdots\right\}$ where $f_n \in V^*$ is determined by $f_n(e_k) = \delta_{n,k}$. Show that $\mathcal{F}$ is linearly independent but does not span V^*.

(d) Define $E : V \to V^{**}$ by $E(a)(f) = f(a)$, where $a \in V$ and $f \in V^*$. Show that E is 1:1 but not onto.

(e) Define $L : V \to V$ by $L(a)_k = \sum_{i=k}^{\infty} a_i$, where $a \in V$. Show that L has no adjoint.