MATH 245 Linear Algebra 2, Assignment 5 Due Mon July 25

1: (a) Let $A = \begin{pmatrix} 3 & 2 & -4 \\ 2 & 0 & -2 \\ -4 & -2 & 3 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^tAP = D$.

(b) Let $A = \begin{pmatrix} 2+i & 1+i \\ i & 3+i \end{pmatrix}$. Find a unitary matrix P and an upper-triangular matrix T so that $P^*AP = T$.

2: Find a singular value decomposition $A = Q\Sigma P^*$ for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \\ 1 & 1 \end{pmatrix}$.

3: A matrix $A \in M_{n\times n}(\mathbf{C})$ is called Hermitian positive-definite when $A^* = A$ and the eigenvalues of A are all positive. Let H_n denote the set of Hermitian positive-definite matrices in $M_{n\times n}(\mathbf{C})$.

(a) Show that every element of H_n has a unique square root in H_n.

(b) Let $A \in H_n$. Show that if $A = Q\Sigma P^*$ is a singular value decomposition of A, then $Q = P$.

4: Let $A \in M_{n\times n}(\mathbf{C})$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A (listed with repetition according to algebraic multiplicity). Show that the following are equivalent.

1. $A\,A^* = A^*A$.
2. $A^* = f(A)$ for some polynomial f.
3. $A^* = AP$ for some unitary matrix P.
4. $\sum_{i,j} \left|A_{i,j}\right|^2 = \sum_i |\lambda_i|^2$.

5: Let $A \in O(3, \mathbf{R})$ and let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be the associated linear map given by $L(x) = Ax$. Show that L is either a rotation, a reflection in some 2-dimensional subspace of $\mathbf{R}^3$, or a rotary inversion (that is a map of the form $-R$ where R is a rotation).