1: (a) For the quadratic curve $7x^2+8xy+y^2+5=0$, find the coordinates of each vertex, find the equation of each asymptote, and sketch the curve.

(b) For the real quadratic form $K(x,y,z)=3x^2+ay^2+bz^2-6xy+2xz-4yz$, sketch the set of points (a,b) for which K is positive-definite.

2: Let U and V be non-trivial subspaces of $\mathbf{R}^n$ with $U\cap V=\{0\}$. Recall that

$$\text{angle}(U,V)=\min\left\{\text{angle}(u,v)\big|0\neq u\in U, 0\neq v\in V\right\}.$$

(a) Show that $\text{angle}(U,V)=\cos^{-1}(\sigma)$ where σ is the largest singular value of the linear map $P:U\to V$ given by $P(x)=\text{Proj}_V(x)$.

(b) Let $u_1=\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}$, $u_2=\begin{pmatrix}2\\1\\-1\\2\end{pmatrix}$, $v_1=\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}$ and $v_2=\begin{pmatrix}2\\1\\1\\0\end{pmatrix}$. Let $U=\text{Span}\{u_1,u_2\}$ and $V=\text{Span}\{v_1,v_2\}$. Find $\text{angle}(U,V)$.

3: Let $\mathbf{F}=\mathbf{Z}_7$, the field of integers modulo 7.

(a) Let $A=\begin{pmatrix}2&1&5\\1&4&3\\5&3&0\end{pmatrix}\in M_{3\times 3}(\mathbf{F})$. Find $Q\in GL(3,\mathbf{F})$ such that $Q^tA\,Q$ is diagonal.

(b) Find the number of distinct congruence classes of 3×3 symmetric matrices over $\mathbf{F}$.

4: (a) Let $A=\begin{pmatrix}1-i&i\\2i&-1+i\end{pmatrix}\in M_{2\times 2}(\mathbf{C})$. Find $\max\limits_{|x|=1}\big|Ax\big|$ and $\min\limits_{|y|=1}\big|Ay\big|$, and find unit vectors x and y for which these maximum and minimum values are attained.

(b) Let $\mathbf{F}=\mathbf{R}$ or $\mathbf{C}$. Let $A\in M_{n\times n}(\mathbf{F})$ with $A^*=A$. Let $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n$ be the eigenvalues of A, listed in increasing order, with repetition according to multiplicity. Show that for each $k=1,2,\cdots,n$ we have

$$\lambda_k=\min_{U\subset\mathbf{F}^n,\dim U=k}\left(\max_{x\in U,|x|=1}x^*A\,x\right)$$

5: Let U and V be vector spaces over a field $\mathbf{F}$ with $\text{char}(\mathbf{F})\neq 2$. For $u\in U$ and $v\in V$, let $u\otimes v$ denote the bilinear map from $U^*\times V^*$ to $\mathbf{F}$ given by

$$(u\otimes v)(f,g)=f(u)g(v)$$

for $f\in U^*$ and $g\in V^*$. For $u,v\in U$ let $u\odot v$ and $u\wedge v$ be the bilinear maps from $U^*\times U^*\to\mathbf{F}$ given by

$$u\odot v=\tfrac{1}{2}\big((u\otimes v)+(v\otimes u)\big)\ ,\quad u\wedge v=\tfrac{1}{2}\big((u\otimes v)-(v\otimes u)\big)\,.$$

Note that $u\odot v$ is symmetric and $u\wedge v$ is alternating. The **tensor product** of U and V is defined to be

$$U\otimes V=\text{Span}\big\{u\otimes v\big|u\in U,v\in V\big\}\subset\text{Bilin}(U^*\times V^*,\mathbf{F})\,.$$

We define the spaces of 2-tensors, symmetric 2-tensors, and alternating 2-tensors on U to be

$$T^2U=U\otimes U$$
$$S^2U=\big\{S\in T^2U\,\big|\,S\text{ is symmetric}\big\}$$
$$\Lambda^2U=\big\{A\in T^2U\,\big|\,A\text{ is alternating}\big\}$$

Suppose that U and V are finite-dimensional, and let $\mathcal{U}=\{u_1,\cdots,u_n\}$ and $\mathcal{V}=\{v_1,\cdots,v_m\}$ be bases.

(a) Show that $\big\{u_i\otimes v_j\big|1\leq i\leq n,1\leq j\leq m\big\}$ is a basis for $U\otimes V$ and that $U\otimes V=\text{Bilin}(U^*\times V^*,\mathbf{F})$.

(b) Show that $\big\{u_i\odot u_j\big|1\leq i\leq j\leq n\big\}$ is a basis for S^2U.

(c) Show that $\big\{u_i\wedge u_j\big|1\leq i<j\leq n\big\}$ is a basis for Λ^2U.