MATH 245 Linear Algebra 2, Solutions to Assignment 1

1: (a) Let A be a set. For each $\alpha \in A$, let $p_\alpha \in \mathbf{R}^n$, let U_α be a vector space in $\mathbf{R}^n$, and let $P = \bigcap_{\alpha\in A} (p_\alpha + U_\alpha)$. Show that if P is not empty then it is an affine space in $\mathbf{R}^n$.

Solution: Let $U = \bigcap_{\alpha\in A} U_\alpha$. We claim that U is a vector space in $\mathbf{R}^n$. Since each U_α is a vector space in $\mathbf{R}^n$, we have $0 \in U_\alpha$ for all $\alpha \in A$, and hence $0 \in U$. Let $u, v \in U$ and let $t \in \mathbf{R}$. Then for all $\alpha \in A$ we have $u \in U_\alpha$ and $v \in U_\alpha$ and so (since U_α is a vector space) we have $tu \in U_\alpha$ and $(u + v) \in U_\alpha$. Since $tu \in U_\alpha$ and $(u + v) \in U_\alpha$ for all $\alpha \in A$, we have $tu \in U$ and $(u + v) \in U$. Thus U is a vector space as claimed.

Suppose that P is not empty and choose $p \in P$. For each $\alpha \in A$, we have $p \in p_\alpha + U_\alpha$ so we can choose $u_\alpha \in U_\alpha$ so that $p = p_\alpha + u_\alpha$. We claim that $P = p+U$. To show that $P \subset p+U$, let $x \in P$. For each $\alpha \in A$, we have $x \in p_\alpha + U_\alpha$ so we can choose $v_\alpha \in U_\alpha$ so that $x = p_\alpha + v_\alpha$. Let $u = x - p$. For each $\alpha \in A$ we have $u = x - p = (p_\alpha + v_\alpha) - (p_\alpha + u_\alpha) = v_\alpha - u_\alpha \in U_\alpha$. Since $u \in U_\alpha$ for all $\alpha \in A$, we have $u \in U$. Hence $x = p+u \in p+U$. Thus $P \subset p+U$. Conversely, to show that $p+U \subset P$, let $y \in p+U$. Choose $w \in U$ so that $y = p + w$. For each $\alpha \in A$ we have $w \in U_\alpha$, so $u_\alpha + w \in U_\alpha$ and hence $y = p + w = p_\alpha + u_\alpha + w \in p_\alpha + U_\alpha$. Since $y \in p_\alpha + U_\alpha$ for all $\alpha \in A$, we have $y \in P$. Thus $p + U \subset P$.

(b) Let $p = \begin{pmatrix}1\\3\\1\\2\\2\end{pmatrix}$, $A = \begin{pmatrix}2&3&1\\1&2&0\\2&4&1\\3&5&2\\1&2&1\end{pmatrix}$, $B = \begin{pmatrix}1&2&1&-1&-2\\1&-3&0&-1&3\end{pmatrix}$ and $q = \begin{pmatrix}1\\-3\end{pmatrix}$, and let $P = p + \text{Col}A$ and $Q = \left\{x \in \mathbf{R}^5 \middle| Bx = q\right\}$. Show that $P \cap Q$ is not empty and find its dimension.

Solution: To have $x \in P \cap Q$, we need $x \in P = p + \text{Col}A$ so we must have $x = p + Ay$ for some $y \in \mathbf{R}^3$, and we need $x \in Q$ so we must have $Bx = q$, that is $B(p + Ay) = q$ or equivalently $BAy = q - Bp$. We solve the equation $BAy = q - Bp$ for y. We have

$$BA = \begin{pmatrix}1&2&1&-1&-2\\1&-3&0&-1&3\end{pmatrix}\begin{pmatrix}2&3&1\\1&2&0\\2&4&1\\3&5&2\\1&2&1\end{pmatrix} = \begin{pmatrix}1&2&-2\\-1&-2&2\end{pmatrix}$$

$$q - Bp = \begin{pmatrix}1\\-3\end{pmatrix} - \begin{pmatrix}1&2&1&-1&-2\\1&-3&0&-1&3\end{pmatrix}\begin{pmatrix}1\\3\\1\\2\\2\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix} - \begin{pmatrix}2\\-4\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

$$\left(BA\middle|q - Bp\right) = \left(\begin{array}{ccc|c}1&2&-2&-1\\-1&-2&2&1\end{array}\right) \sim \left(\begin{array}{ccc|c}1&2&-2&-1\\0&0&0&0\end{array}\right)$$

$$y = \begin{pmatrix}-1\\0\\0\end{pmatrix} + s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$x = p + Ay = \begin{pmatrix}1\\3\\1\\2\\2\end{pmatrix} + \begin{pmatrix}2&3&1\\1&2&0\\2&4&1\\3&5&2\\1&2&1\end{pmatrix}\left(\begin{pmatrix}-1\\0\\0\end{pmatrix} + s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\1\end{pmatrix}\right) = \begin{pmatrix}-1\\2\\-1\\-1\\1\end{pmatrix} + s\begin{pmatrix}-1\\0\\0\\-1\\0\end{pmatrix} + t\begin{pmatrix}5\\2\\5\\8\\3\end{pmatrix}.$$

Thus $P \cap Q$ contains the point $(-1, 2, -1, -1, 1)^t$ and is 2-dimensional.

2: For two affine spaces P and Q in $\mathbf{R}^n$, the **distance** between P and Q is defined to be

$$\text{dist}(P,Q) = \min\left\{\text{dist}(x,y)\Big|x\in P, y\in Q\right\}.$$

(a) Let p and q be points in $\mathbf{R}^n$, let U and V be subspaces of $\mathbf{R}^n$, and let $P = p+U$ and $Q = q+V$. Show that

$$\text{dist}(P,Q) = \left|\text{Proj}_{(U+V)^\perp}(p-q)\right|.$$

Solution: We have

$$\begin{aligned}
\text{dist}(P,Q) &= \min\left\{\text{dist}(x,y)\big|x\in P, y\in Q\right\}\\
&= \min\left\{\text{dist}(p+u,q+v)\big|u\in U, v\in V\right\}\\
&= \min\left\{\left|(q+v)-(p+u)\right|\big|u\in U, v\in V\right\}\\
&= \min\left\{\left|(q-p)-(u-v)\right|\big|u\in U, v\in V\right\}\\
&= \min\left\{\left|(q-p)-w\right|\big|w\in U+V\right\}\\
&= \left|(q-p)-\text{Proj}_{U+V}(q-p)\right|\\
&= \left|\text{Proj}_{(U+V)^\perp}(q-p)\right|
\end{aligned}$$

where, on the second last line, we used the fact that $\text{Proj}_{U+V}(q-p)$ is the (unique) point on $U+V$ which is nearest to $q-p$.

(b) Let $p=(1,2,4,3)^t$, $u_1=(1,2,0,1)^t$, $u_2=(3,5,1,2)^t$, $q=(4,3,1,2)^t$, $v_1=(2,3,2,-1)^t$, $v_2=(1,3,1,-2)^t$. Find the distance between the plane $x=p+t_1u_1+t_2u_2$ and the plane $x=q+s_1v_1+s_2v_2$.

Solution: Let $A=(u_1,u_2)\in M_{4\times 2}$ and $B=(v_1,v_2)\in M_{4\times 2}$, and let $U=\text{Col}(A)$ and $V=\text{Col}(B)$, so that the given two planes are $P=p+U$ and $Q=q+V$. Note that $(U+V)^\perp = (\text{Col}A+\text{Col}B)^\perp = \text{Col}(A,B)^\perp = \text{Null}(C)$ where $C=(A,B)^t$. We have

$$C = \begin{pmatrix}1&2&0&1\\3&5&1&2\\2&3&2&-1\\1&3&1&-2\end{pmatrix} \sim \begin{pmatrix}1&2&0&1\\0&1&-1&1\\0&1&-2&3\\0&1&1&-1\end{pmatrix} \sim \begin{pmatrix}1&0&2&-1\\0&1&-1&1\\0&0&1&-2\\0&0&2&-4\end{pmatrix} \sim \begin{pmatrix}1&0&0&3\\0&1&0&-1\\0&0&1&-2\\0&0&0&0\end{pmatrix}$$

so that $(U+V)^\perp = \text{Null}(C) = \text{Span}\{w\}$ where $w=\big(-3,1,2,1\big)^t$. Writing $x=q-p=(3,1,-3,-1)^t$, we have

$$\text{Proj}_{(U+V)^\perp}(q-p) = \text{Proj}_w(x) = \frac{w\cdot x}{|w|^2}\,w = \frac{-15}{15}\begin{pmatrix}-3\\1\\2\\1\end{pmatrix} = \begin{pmatrix}3\\-1\\-2\\-1\end{pmatrix}$$

and so

$$\text{dist}(P,Q) = \left|\text{Proj}_{(U+V)^\perp}(q-p)\right| = \sqrt{15}\,.$$

3: For two non-trivial vector spaces U and V in $\mathbf{R}^n$, we define the **angle** between U and V, which we write as $\text{angle}(U,V)$, as follows. If $U \subset V$ or $V \subset U$ then $\text{angle}(U,V) = 0$, otherwise if $U \cap V = \{0\}$ then

$$\text{angle}(U,V) = \min\left\{\theta(u,v)\Big|0 \neq u \in U, 0 \neq v \in V\right\},$$

and if $U \cap V = W \neq \{0\}$ then $\text{angle}(U,V) = \text{angle}(U \cap W^\perp, V \cap W^\perp)$. We define the angle between two affine spaces in $\mathbf{R}^n$ to be the angle between their associated vector spaces.

(a) Let $0 \neq u \in \mathbf{R}^n$, let $U = \text{Span}\{u\}$, and let V be a non-trivial vector space in $\mathbf{R}^n$. Show that

$$\text{angle}(U,V) = \cos^{-1}\left|\text{Proj}_V\left(\frac{u}{|u|}\right)\right|.$$

Solution: Suppose first that $u \in V$. On the one hand, since $U \subset V$, we have $\text{angle}(U,V) = 0$, and on the other hand, we have $\text{Proj}_V \frac{u}{|u|} = \frac{u}{|u|}$ so $\cos^{-1}\left|\text{Proj}_V \frac{u}{|u|}\right| = \cos^{-1}(1) = 0$. Thus $\text{angle}(U,V) = 0 = \cos^{-1}\left|\text{Proj}_V \frac{u}{|u|}\right|$.

Next, suppose that $u \in V^\perp$. On the one hand, we have $tu \bullet v = 0$ for all $t \in \mathbf{R}$ and all $v \in V$, and so $\theta(tu,v) = \frac{\pi}{2}$ for all $0 \neq t \in \mathbf{R}$ and all $0 \neq v \in V$, and on the other hand we have $\text{Proj}_V \frac{u}{|u|} = 0$ so $\cos^{-1}\left|\text{Proj}_V \frac{u}{|u|}\right| = \cos^{-1}(0) = \frac{\pi}{2}$. Thus $\text{angle}(U,V) = \frac{\pi}{2} = \cos^{-1}\left|\text{Proj}_V \frac{u}{|u|}\right|$.

Finally, suppose that $u \notin V$ and $u \notin V^\perp$. Let $v = \text{Proj}_V(u)$ and let $\theta = \theta(u,v)$. Note that $v \neq 0$ (since $u \notin V^\perp$) and $v \neq u$ (since $u \notin V$). Using trigonometric ratios (for the triangle $[0,v,u]$) we have

$$\cos\theta = \frac{|v|}{|u|} = \tfrac{1}{|u|}\left|\text{Proj}_V(u)\right| = \left|\text{Proj}_V \tfrac{u}{|u|}\right|.$$

Thus we must show that $\text{angle}(U,V) = \theta$. Equivalently, we must show that

$$\theta = \theta(u,v) \leq \theta(tu,w) \text{ for all } 0 \neq t \in \mathbf{R}, 0 \neq w \in V.$$

First we claim that $0 < \theta < \frac{\pi}{2}$. Since $v \neq 0$ we have $\cos\theta = \frac{|v|}{|u|} > 0$. Since $v \neq u$ so that $|v-u| \neq 0$, by Pythagoras' Theorem we have $|u|^2 = |v|^2 + |v-u|^2 > |v|^2$ so that $|u| > |v|$, and so $\cos\theta = \frac{|v|}{|u|} < 1$. Since $0 < \cos\theta < 1$ we have $0 < \theta < \frac{\pi}{2}$, as claimed.

Let $0 \neq t \in \mathbf{R}$ and let $0 \neq w \in V$. Suppose first that $w \in \text{Span}\{v\}$, say $w = sv$ with $0 \neq s \in \mathbf{R}$. Then we have

$$\theta(tu,w) = \theta(tu,sv) = \begin{cases} \theta(u,v) & \text{if } st > 0 \\ \pi - \theta(u,v) & \text{if } st < 0 \end{cases} = \begin{cases} \theta & \text{if } st > 0 \\ \pi - \theta & \text{if } st < 0 \end{cases}$$

Since $\theta \in \left(0, \frac{\pi}{2}\right)$, we have $\pi - \theta \in \left(\frac{\pi}{2}, \pi\right)$, so $\pi - \theta > \theta$, and hence $\theta(tu,w) \geq \theta = \theta(u,v)$.

Now suppose that $w \notin \text{Span}\{v\}$. Let $y = \text{Proj}_w(u)$. Note that if $y = 0$ then $tu \bullet w = u \bullet w = 0$ so we have $\theta(tu,w) = \frac{\pi}{2} > \theta$. Suppose that $y \neq 0$. As above (where we showed that $0 < \theta < \frac{\pi}{2}$) we have $0 < \theta(u,y) < \frac{\pi}{2}$. Since v is the point in V nearest to u, we know that $|u-y| > |u-v|$, so using trigonometric ratios (for the triangle $[0,y,u]$) gives

$$\sin\big(\theta(u,y)\big) = \frac{|u-y|}{|u|} > \frac{|u-v|}{|u|} = \sin\big(\theta(u,v)\big).$$

Thus $0 < \theta(u,v) < \theta(u,y) < \frac{\pi}{2}$. Note that $w \in \text{Span}\{y\}$, say $w = sy$ where $0 \neq s \in \mathbf{R}$. When $st > 0$ we have $\theta(tu,w) = \theta(tu,sy) = \theta(u,y) > \theta(u,v)$, and when $st < 0$ we have $\theta(tu,w) = \theta(tu,sy) = \pi - \theta(u,y) > \pi - \frac{\pi}{2} = \frac{\pi}{2} > \theta(u,v)$.

(b) Let $u_1=(1,-2,1,-3)^t$, $u_2=(3,2,1,-1)^t$, $v_1=(1,-1,0,1)^t$, $v_2=(1,-3,2,-1)^t$ and $v_3=(2,-1,1,-1)^t$. Find the angle between $U=\text{Span}\{u_1,u_2\}$ and $V=\text{Span}\{v_1,v_2,v_3\}$.

Solution: Let $A=(u_1,u_2)\in M_{4\times 2}$ and $B=(v_1,v_2,v_3)\in M_{4\times 3}$ so that $U=\text{Col}(A)$ and $V=\text{Col}(B)$. Let us find a basis for $V^\perp=\text{Null}(B^t)$. We have

$$B^t=\begin{pmatrix}1&-1&0&1\\1&-3&2&-1\\2&-1&1&-1\end{pmatrix}\sim\begin{pmatrix}1&-1&0&1\\0&2&-2&2\\0&1&1&-3\end{pmatrix}\sim\begin{pmatrix}1&0&1&-2\\0&1&1&-3\\0&0&4&-8\end{pmatrix}\sim\begin{pmatrix}1&0&1&-2\\0&1&1&-3\\0&0&1&-2\end{pmatrix}\sim\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&-2\end{pmatrix}$$

so $V^\perp$ has basis $\{(0,1,2,1)^t\}$. Thus $V=\text{Null}(C)$ where $C=\begin{pmatrix}0&1&2&1\end{pmatrix}\in M_{1\times 4}$.

Now let us find a basis for $W=U\cap V=\text{Col}(A)\cap\text{Null}(C)$. To have $x\in W$ we need $x\in\text{Col}(A)$, say $x=At$, and we need $x\in\text{Null}(C)$, that is $0=Cx=CAt$, so $t\in\text{Null}(CA)$. We have

$$CA=\begin{pmatrix}0&1&2&1\end{pmatrix}\begin{pmatrix}1&3\\-2&2\\1&1\\-3&-1\end{pmatrix}=\begin{pmatrix}-3&3\end{pmatrix}\sim\begin{pmatrix}1&-1\end{pmatrix}$$

so we have $\text{Null}(CA)=\text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ and $W=\text{Span}\left\{A\begin{pmatrix}1\\1\end{pmatrix}\right\}$. Note that

$$A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1&3\\-2&2\\1&1\\-3&-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\0\\2\\-4\end{pmatrix}=2\begin{pmatrix}2\\0\\1\\-2\end{pmatrix},$$

so we have $W=\text{Span}\{(2,0,1,-2)^t\}$ and hence $W^\perp=\text{Null}(D)$ where $D=\begin{pmatrix}2&0&1&-2\end{pmatrix}$.

Next, consider $U\cap W^\perp$ and $V\cap W^\perp$. To have $x\in U\cap W^\perp=\text{Col}(A)\cap\text{Null}(D)$, we need $x\in\text{Col}(A)$, say $x=At$, and we need $x\in\text{Null}(D)$ so $0=Dx=DAt$. We have

$$DA=\begin{pmatrix}2&0&1&-2\end{pmatrix}\begin{pmatrix}1&3\\-2&2\\1&1\\-3&-1\end{pmatrix}=\begin{pmatrix}9&9\end{pmatrix}\sim\begin{pmatrix}1&1\end{pmatrix}$$

so $\text{Null}(DA)=\text{Span}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$ and $U\cap W^\perp=\text{Span}\left\{A\begin{pmatrix}-1\\1\end{pmatrix}\right\}$. Note that

$$A\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1&3\\-2&2\\1&1\\-3&-1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}2\\4\\0\\2\end{pmatrix}=2\begin{pmatrix}1\\2\\0\\1\end{pmatrix},$$

so $U\cap W^\perp=\text{Span}\{u\}$ where $u=(1,2,0,1)^t$. Also, $V\cap W^\perp=\text{Null}(C)\cap\text{Null}(D)=\text{Null}\begin{pmatrix}C\\D\end{pmatrix}$, so we have

$$(V\cap W^\perp)^\perp=\text{Col}(E),\text{ where } E=\begin{pmatrix}C\\D\end{pmatrix}^t=\begin{pmatrix}0&2\\1&0\\2&1\\1&-2\end{pmatrix}.$$

Since $E^tE=\begin{pmatrix}0&1&2&1\\2&0&1&-2\end{pmatrix}\begin{pmatrix}0&2\\1&0\\2&1\\1&-2\end{pmatrix}=\begin{pmatrix}6&0\\0&6\end{pmatrix}=6I$, we have

$$\text{Proj}_{(V\cap W^\perp)^\perp}(u)=E(E^tE)^{-1}E^tu=\begin{pmatrix}0&2\\1&0\\2&1\\1&-2\end{pmatrix}\cdot\tfrac{1}{6}I\cdot\begin{pmatrix}0&1&2&1\\2&0&1&-2\end{pmatrix}\begin{pmatrix}1\\2\\0\\1\end{pmatrix}=\tfrac{1}{6}\begin{pmatrix}0&2\\1&0\\2&1\\1&-2\end{pmatrix}\begin{pmatrix}3\\0\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}0\\1\\2\\1\end{pmatrix},$$

$$\text{Proj}_{V\cap W^\perp}(u)=u-\text{Proj}_{(V\cap W^\perp)^\perp}(u)=\begin{pmatrix}1\\2\\1\\1\end{pmatrix}-\tfrac{1}{2}\begin{pmatrix}0\\1\\2\\1\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}2\\3\\-2\\1\end{pmatrix},\text{ and}$$

$$\text{angle}(U,V)=\cos^{-1}\frac{\left|\text{Proj}_{V\cap W^\perp}(u)\right|}{|u|}=\cos^{-1}\frac{\frac{1}{2}\sqrt{18}}{\sqrt{6}}=\cos^{-1}\tfrac{\sqrt{3}}{2}=\tfrac{\pi}{6}.$$

4: Let $a_0, a_1, \cdots, a_l$ be points in $\mathbf{R}^n$. Show that $[a_0, a_1, \cdots, a_l] = \left\{ \sum\limits_{i=0}^{l} s_i a_i \middle| 0 \leq s_i \in \mathbf{R}, \sum\limits_{i=0}^{l} s_i = 1 \right\}$.

Solution: Let $S = \left\{ \sum\limits_{i=0}^{l} s_i a_i \middle| 0 \leq s_i \in \mathbf{R}, \sum\limits_{i=0}^{l} s_i = 1 \right\}$. Note that each $a_k \in S$. We claim that S is convex. Let $x, y \in S$, say $x = \sum\limits_{i=0}^{l} s_i a_i$ and $y = \sum\limits_{i=0}^{l} t_i a_i$ where $0 \leq s_i, t_i$ and $\sum s_i = \sum t_i = 1$. Let $z \in [x, y]$, say $z = x + r(y - x)$ where $0 \leq r \leq 1$. Then we have $z = \sum s_i a_i + r\left(\sum t_i a_i - \sum s_i a_i\right) = \sum r_i a_i$ where $r_i = s_i + r(t_i - s_i)$ for $i = 0, 1, \cdots, l$. Since $0 \leq s_i$ and $0 \leq t_i$ and $r_i \in [s_i, t_i]$ (so r_i is between s_i and t_i), we must have $r_i \geq 0$. Also, we have $\sum r_i = \sum s_i + r\left(\sum t_i - \sum s_i\right) = 1 + r(1 - 1) = 1$, and so $z = \sum r_i a_i \in S$. Thus S is convex, as claimed. Since S is a convex set which contains all the points $a_0, a_1, \cdots, a_l$, we have $[a_0, a_1, \cdots, a_l] \subset S$.

To show that $S \subset [a_0, a_1, \cdots, a_l]$, we shall show that S is contained in every convex set which contains $a_0, a_1, \cdots, a_l$. Let T be a convex set with $a_i \in T$ for all $i = 0, 1, \cdots, l$. For each $k = 0, 1, \cdots, l$, let $S_k = \left\{ \sum\limits_{i=0}^{k} s_i a_i \middle| 0 \leq s_i \in \mathbf{R}, \sum\limits_{i=0}^{k} s_i = 1 \right\}$. We claim that each $S_k \subset T$ (and in particular, $S = S_l \subset T$). We have $S_0 = \{a_0\} \subset T$. Fix $k \geq 1$ and suppose, inductively, that $S_{k-1} \subset T$. Let $x \in S_k$, say $s = \sum\limits_{i=0}^{k} s_i a_i$ with $0 \leq s_i$, $\sum s_i = 1$. If $s_k = 1$ then $x = a_k$ and so $x \in T$. Suppose that $s_k \neq 1$. Let $y = \sum\limits_{i=0}^{k-1} \frac{s_i}{1-s_k} a_i$. Note that each $\frac{s_i}{1-s_k} \geq 0$ and that $\sum\limits_{i=0}^{k-1} \frac{s_i}{1-s_k} = \frac{1}{1-s_k} \sum\limits_{i=0}^{k-1} s_i = \frac{1}{1-s_k}(1 - s_k) = 1$ and so we have $y \in S_{k-1} \subset T$. Also, we have $(1 - s_k)y = \sum\limits_{i=0}^{k-1} s_i a_i = x - s_k a_k$ and so $x = (1 - s_k)y + s_k a_k = y + s_k(a_k - y) \in [y, a_k]$. Since $y \in S_{k-1} \subset T$ and $a_k \in T$ and T is convex, we have $x \in T$. Thus $S_k \subset T$. By induction, we have $S_k \subset T$ for all $k = 0, 1, \cdots, l$, and in particular $S = S_l \subset T$.

5: Let $S=\big[a_0,a_1,\cdots,a_l\big]$ be an l-simplex in $\mathbf{R}^n$. for each $0\le j<k\le n$, the **altitudinal hyperplane** $H_{j,k}$ is the $(l-1)$-dimensional affine space in $\langle a_0,a_1,\cdots,a_l\rangle$ which is perpendicular to the edge $[a_j,a_k]$ and which passes through the centroid of the $(l-2)$-simplex $\big[a_0,a_1,\cdots,\check{a}_j,\cdots,\check{a}_k,\cdots,a_l\big]$, (where the check mark above the points a_j and a_k indicates that these points are excluded). Show that the altitudinal hyperplanes have a unique point of intersection. This point is denoted by h and is called the **orthocenter** of the l-simplex S.

Solution: Let $u_k=a_k-a_0$ for $k=1,2,\cdots,l$, and let $A=\big(u_1,u_2,\cdots,u_l\big)$. To have $x\in\langle a_0,a_1,\cdots,a_l\rangle$ we need $x=a_0+Ay$ for some $y\in\mathbf{R}^l$. Let $g_{j,k}$ denote the centroid of the simplex $[a_0,\cdots\check{a}_j,\cdots,\check{a}_k,\cdots,a_n]$. Note that

$$g_{0,k}=\tfrac{1}{l-1}\textstyle\sum_{i\neq 0,k}a_i=\tfrac{1}{l-1}\left(\sum\limits_{i=1}^{l}a_i-a_k\right)=a_0+\tfrac{1}{l-1}\left(\sum\limits_{i=1}^{l}u_i-u_k\right)=a_0+\tfrac{1}{l-1}\big(Ac-u_k\big)$$

where c is the vector $c=(1,1,\cdots,1)^t$, and so for each $k=1,2,\cdots,l$ we have

$$\begin{aligned}x\in H_{0,k}&\iff (x-g_{0,k})\bullet(a_k-a_0)=0\\&\iff \big((a_0+Ay)-\big(a_0+\tfrac{1}{l-1}(Ac-u_k)\big)\big)\bullet u_k=0\\&\iff Ay\bullet u_k=\tfrac{1}{l-1}(Ac-u_k)\bullet u_k\end{aligned}$$

and hence

$$x\in\bigcap_{k=1}^{l}H_{0,k}\iff A^tAy=\tfrac{1}{l-1}\big(A^tAc-v\big)$$

where v is the vector $v=\big(|u_1|^2,|u_2|^2,\cdots,|u_l|^2\big)^t$. Since the simplex S is non-degenerate, $\{u_1,u_2,\cdots,u_l\}$ is linearly independent, so $\text{rank}(A^tA)=\text{rank}(A)=l$ and hence A^tA is invertible. Thus

$$x\in\bigcap_{k=1}^{l}H_{0,k}\iff y=\tfrac{1}{l-1}\big(c-(A^tA)^{-1}v\big)\,.$$

This shows that the altitudinal hyperplanes $H_{0,k}$ for $k=1,2,\cdots,l$ have a unique point of intersection, namely the point $x=a_0+Ay$ where $y=\frac{1}{l-1}\big(c-(A^tA)^{-1}v\big)$.

It remains to show that the above point x lies on all the altitudinal hyperplanes $H_{j,k}$ for $1\le j<k\le l$. Let $1\le j<k\le l$. Note that

$$g_{j,k}=\tfrac{1}{l-1}\sum_{i\neq j,k}a_i=\tfrac{1}{l-1}\Big(\sum_{i\neq 0,k}a_i-(a_j-a_0)\Big)=g_{0,k}-\tfrac{1}{l-1}u_j\,.$$

Similarly $g_{j,k}=g_{0,j}-\frac{1}{l-1}u_k$. Since $x\in H_{0,k}$ we have $(x-g_{0,k})\bullet u_k=0$, and since $x\in H_{0,j}$ we have $(x-g_{0,j})\bullet u_j=0$, and so

$$\begin{aligned}(x-g_{j,k})\bullet(a_k-a_j)&=(x-g_{j,k})\bullet(u_k-u_j)\\&=(x-g_{j,k})\bullet u_k-(x-g_{j,k})\bullet u_j\\&=\big(x-g_{0,k}+\tfrac{1}{l-1}u_j\big)\bullet u_k-\big(x-g_{0,j}+\tfrac{1}{l-1}u_k\big)\bullet u_j\\&=(x-g_{0,k})\bullet u_k+\tfrac{1}{l-1}u_j\bullet u_k-(x-g_{0,j})\bullet u_j-\tfrac{1}{l-1}u_k\bullet u_j\\&=0\end{aligned}$$

hence $x\in H_{j,k}$, as required.

(b) Let $S=\big[a_0,a_1,\cdots,a_l\big]$ be an l-simplex in $\mathbf{R}^n$. Let o, g and h be the circumcenter, the centroid, and the orthocenter of S. Show that g lies $\frac{l-1}{l+1}$ of the way along the line segment from o to h.

Solution: We know from class that $o=a_0+At$ where $t=\frac{1}{2}(A^tA)^{-1}v$ with $v=\big(|u_1|^2,\cdots,|u_l|^2\big)^t$, and we know from part (a) that $h=a_0+Ay$ where $y=\frac{1}{l-1}(c-(A^tA)^{-1}v)$ with $c=(1,1,\cdots,1)^t$, and so the point which lies $\frac{l-1}{l+1}$ of the way from o to h is the point

$$\begin{aligned}o+\tfrac{l-1}{l+1}(h-o)&=a_0+At+\tfrac{l-1}{l+1}(Ay-At)=a_0+\tfrac{2}{l+1}At-\tfrac{l-1}{l+1}Ay\\&=a_0+\tfrac{1}{l+1}A(A^tA)^{-1}v+\tfrac{l}{l+1}\big(A\big(c-(A^tA)^{-1}v\big)\\&=a_0+\tfrac{1}{l+1}Ac=a_0+\tfrac{1}{l+1}\sum_{i=1}^{l}u_i=a_0+\tfrac{1}{l+1}\sum_{i=1}^{l}(a_i-a_0)\\&=\tfrac{1}{l+1}\sum_{i=0}^{l}a_i\,,\end{aligned}$$

which is the centroid of S, as required.