MATH 245 Linear Algebra 2, Solutions to Assignment 2

1: Let $p_1(x)=x-1$, $p_2(x)=\frac{1}{2}(x^2-3x)$ and $p_3(x)=\frac{1}{2}(x^3-3x^2+2)$. Find the polynomial $f\in\text{Span}\{p_1,p_2,p_3\}$ which minimizes the sum $\sum_{i=1}^{5}\big(f(a_i)-b_i\big)^2$ for the 5 points (a_i,b_i) given below

i	1	2	3	4	5
a_i	-1	0	1	2	3
b_i	-2	1	2	0	1

Solution: We want to minimize the distance between $\big(f(a_1),f(a_2),\cdots,f(a_5)\big)^t$ and $b=(b_1,b_2,\cdots,b_5)^t$. We have

$$\begin{pmatrix}f(a_1)\\f(a_2)\\f(a_3)\\f(a_4)\\f(a_5)\end{pmatrix}=\begin{pmatrix}c_1\,p_1(a_1)+c_2\,p_2(a_1)+c_3\,p_3(a_1)\\c_1\,p_1(a_2)+c_2\,p_2(a_2)+c_3\,p_3(a_2)\\c_1\,p_1(a_3)+c_2\,p_2(a_3)+c_3\,p_3(a_3)\\c_1\,p_1(a_4)+c_2\,p_2(a_4)+c_3\,p_3(a_4)\\c_1\,p_1(a_5)+c_2\,p_2(a_5)+c_3\,p_3(a_5)\end{pmatrix}=Ac$$

where

$$A=\begin{pmatrix}p_1(a_1)&p_2(a_1)&p_3(a_1)\\p_1(a_2)&p_2(a_2)&p_3(a_2)\\p_1(a_3)&p_2(a_3)&p_3(a_3)\\p_1(a_4)&p_2(a_4)&p_3(a_4)\\p_1(a_5)&p_2(a_5)&p_3(a_5)\end{pmatrix}=\begin{pmatrix}-2&2&-1\\-1&0&1\\0&-1&0\\1&-1&-1\\2&0&1\end{pmatrix}$$

and $c=(c_1,c_2,c_3)^t$. To minimize the distance between Ac and b we must choose c so that $Ac=\text{Proj}_{\text{Col}A}(b)$. Writing $u=Ac=\text{Proj}_{\text{Col}A}(b)$ and $v=b-Ac=\text{Proj}_{(\text{Col}A)^\perp}(b)$ we have $u+v=b$, that is $Ac+v=b$, and so $A^tAc=A^tb$. We solve the equation $A^tAc=A^tb$ for c. We have

$$A^tA=\begin{pmatrix}-2&-1&0&1&2\\2&0&-1&-1&0\\-1&1&0&-1&1\end{pmatrix}\begin{pmatrix}-2&2&-1\\-1&0&1\\0&-1&0\\1&-1&-1\\2&0&1\end{pmatrix}=\begin{pmatrix}10&-5&2\\-5&6&-1\\2&-1&4\end{pmatrix}$$

$$A^tb=\begin{pmatrix}-2&-1&0&1&2\\2&0&-1&-1&0\\-1&1&0&-1&1\end{pmatrix}\begin{pmatrix}-2\\1\\2\\0\\1\end{pmatrix}=\begin{pmatrix}5\\-6\\4\end{pmatrix}$$

$$\big(A^tA\big|A^tb\big)=\left(\begin{array}{ccc|c}10&-5&2&5\\-5&6&-1&-6\\2&1&4&4\end{array}\right)\sim\left(\begin{array}{ccc|c}0&7&0&-7\\1&3&11&6\\2&-1&4&4\end{array}\right)\sim\left(\begin{array}{ccc|c}1&3&11&6\\0&1&0&-1\\0&7&18&8\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&0&11&9\\0&1&0&-1\\0&0&18&15\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&11&9\\0&1&0&-1\\0&0&1&\frac{5}{6}\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&-\frac{1}{6}\\0&1&0&-1\\0&0&1&\frac{5}{6}\end{array}\right)$$

Thus $c=\left(-\frac{1}{6},-1,\frac{5}{6}\right)^t$ and so

$$f(x)=-\tfrac{1}{6}(x-1)-\tfrac{1}{2}(x^2-3x)+\tfrac{5}{12}(x^3-3x^2+2)=\tfrac{5}{12}x^3-\tfrac{7}{4}x^2+\tfrac{4}{3}x+1\,.$$

2: (a) Find the perimeter of the regular hexagon on $\mathbf{S}^2$ with interior angles equal to $\frac{5\pi}{6}$.

Solution: Let u be the center of the hexagon, and let v and w be vertices of the hexagon, with w next to v, counterclockwise. Note that the hexagon can be divided into 6 triangles meeting at u, each of which is congruent to the triangle $[u,v,w]$. For this triangle, we have $\alpha=\frac{\pi}{3}$, $\beta=\gamma=\frac{5\pi}{12}$, and $a=\text{dist}(u,v)$. The perimeter L of the hexagon is given by $L=6a$. Using the Second Law of Cosines we have

$$\cos a=\frac{\cos\alpha+\cos\beta\cos\gamma}{\sin\beta\sin\gamma}=\frac{\cos\frac{\pi}{3}+\cos^2\frac{5\pi}{12}}{\sin^2\frac{5\pi}{12}}\,.$$

Note that $\cos^2\frac{5\pi}{12}=\frac{1+\cos\frac{5\pi}{6}}{2}=\frac{1-\frac{\sqrt{3}}{2}}{2}=\frac{2-\sqrt{3}}{4}$ and $\sin^2\frac{5\pi}{3}=\frac{1-\cos\frac{5\pi}{6}}{2}=\frac{1+\frac{\sqrt{3}}{2}}{2}=\frac{2+\sqrt{3}}{4}$, so

$$\cos a=\frac{\frac{1}{2}+\frac{2-\sqrt{3}}{4}}{\frac{2+\sqrt{3}}{4}}=\frac{4-\sqrt{3}}{2+\sqrt{3}}=(4-\sqrt{3})(2-\sqrt{3})=11-6\sqrt{3}\,,$$

and so the perimeter is $L=6a=6\cos^{-1}\big(11-6\sqrt{3}\big)$.

(b) Find the area of the regular hexagon on $\mathbf{S}^2$ with sides of length $\frac{\pi}{6}$.

Solution: Again, we let u be the center of the hexagon, and let v and w be vertices of the hexagon, with w next to v, counterclockwise. In the triangle $[u,v,w]$ we have $\alpha=\frac{\pi}{3}$, $a=\frac{\pi}{6}$ and $\beta=\gamma$. Writing $\theta=\beta=\gamma$, the Second Law of Cosines gives

$$\begin{aligned}\cos\tfrac{\pi}{6}&=\frac{\cos\frac{\pi}{3}+\cos^2\theta}{\sin^2\theta}\\ \tfrac{\sqrt{3}}{2}\sin^2\theta&=\tfrac{1}{2}+\cos^2\theta\\ \sqrt{3}(1-\cos^2\theta)&=1+2\cos^2\theta\\ (2-\sqrt{3})\cos^2\theta&=\sqrt{3}-1\\ \cos^2\theta=\tfrac{\sqrt{3}-1}{2+\sqrt{3}}&=(\sqrt{3}-1)(2-\sqrt{3})=3\sqrt{3}-5\,.\end{aligned}$$

Thus $\theta=\cos^{-1}\left(\sqrt{3\sqrt{3}-5}\right)$ and the area of the hexagon is

$$A=6\big(\alpha+\beta+\gamma-\pi\big)=6\left(\tfrac{\pi}{3}+2\,\theta-\pi\right)=12\,\theta-4\pi=12\left(\cos^{-1}\sqrt{3\sqrt{3}-5}\right)-4\pi\,.$$

3: (a) Let $u = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $v = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $w = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$. Find the area of the triangle T on $\mathbf{S}^2$ given by

$$T = \left\{x \in \mathbf{S}^2 \,\middle|\, \text{dist}(x,u) \leq \tfrac{\pi}{2},\ \text{dist}(x,v) \leq \tfrac{\pi}{2} \text{ and } \text{dist}(x,w) \leq \tfrac{\pi}{2}\right\}.$$

Solution: Notice that T is the polar triangle of $[u,v,w]$, that is $T = \big[u',v',w'\big]$. In triangle $[u,v,w]$ we have $a = \cos^{-1}(v \bullet w) = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$, $b = \cos^{-1}(w \bullet u) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ and $c = \cos^{-1}(u \bullet v) = \cos^{-1}(0) = \frac{\pi}{2}$ and so in the polar triangle $T = [u',v',w']$ we have $\alpha' = \pi - a = \frac{2\pi}{3}$, $\beta' = \pi - b = \frac{\pi}{3}$ and $\gamma' = \pi - c = \frac{\pi}{2}$. Thus the area of T is

$$A = \alpha' + \beta' + \gamma' - \pi = \tfrac{2\pi}{3} + \tfrac{\pi}{3} + \tfrac{\pi}{2} - \pi = \tfrac{\pi}{2}\,.$$

(b) Let $u = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\-1\end{pmatrix}$, $v = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $w = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}$. Find the circumcenter of triangle $[u,v,w]$ on $\mathbf{S}^2$.

Solution: Note that

$$\begin{aligned}
\text{dist}(x,u) = \text{dist}(x,v) = \text{dist}(x,w)& \\
&\iff \cos^{-1} x \bullet u = \cos^{-1} x \bullet v = \cos^{-1} x \bullet w \\
&\iff x \bullet u = x \bullet v = x \bullet w \\
&\iff x \bullet u = x \bullet v \text{ and } x \bullet u = x \bullet w \\
&\iff x \bullet (u - v) = 0 \text{ and } x \bullet (u - w) = 0 \\
&\iff x \bullet (1,-1,-2)^t = 0 \text{ and } x \bullet (1,2,-3)^t = 0 \\
&\iff x_1 - x_2 - 2x_3 = 0 \text{ and } x_1 + 2x_2 - 3x_3 = 0\,.,
\end{aligned}$$

We solve these two equations. We have

$$\begin{pmatrix}1 & -1 & -2\\ 1 & 2 & -3\end{pmatrix} \sim \begin{pmatrix}1 & -1 & -2\\ 0 & 3 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -\frac{7}{3}\\ 0 & 1 & -\frac{1}{3}\end{pmatrix}$$

so the solution is $x = s\,(7,1,3)^t$ for $s \in \mathbf{R}$. To get $|x|^2 = 1$, we need $s^2(7^2 + 1^2 + 3^2) = 1$ and so we must use $s = \pm\frac{1}{\sqrt{59}}$. We choose the positive square root and obtain

$$x = \tfrac{1}{\sqrt{59}}\begin{pmatrix}7\\1\\3\end{pmatrix}.$$

4: (a) Let R be the radius of the Earth, in meters ($R \cong 6,370,000$). We describe the position of a point on the Earth in terms of its longitude θ (with $\theta = 0$ at Greenwitch, England and $\theta = \frac{\pi}{2}$ somewhere in Bangladesh) and its latitude ϕ (with $\phi = 0$ at the equator and $\phi = \frac{\pi}{2}$ at the north pole). Find the distance (expressed as a multiple of R) and the bearing (expressed as an angle north of east) from the point at $(\theta, \phi) = (\frac{\pi}{3}, \frac{\pi}{6})$ to the point at $(\theta, \phi) = (\frac{\pi}{2}, \frac{\pi}{4})$.

Solution: Consider the spherical triangle with vertices at u, v and w where u is given by $(\theta, \phi) = \left(\frac{\pi}{3}, \frac{\pi}{6}\right)$, v is given by $(\theta, \phi) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$, and w is the north pole, which is given by $\phi = \frac{\pi}{2}$. For this triangle we have $a = R \cdot \frac{\pi}{4}$, $b = R \cdot \frac{\pi}{3}$, and $\gamma = \frac{\pi}{6}$. The First Law of Cosines, modified for a sphere of radius R, gives

$$\cos\gamma = \frac{\cos(c/R) - \cos(b/R)\cos(a/R)}{\sin(b/R)\sin(a/R)}$$

so

$$\begin{aligned} \cos(c/R) &= \cos\gamma \sin(b/R)\sin(a/R) + \cos(b/R)\cos(a/R) \\ &= \cos\tfrac{\pi}{6}\sin\tfrac{\pi}{4}\sin\tfrac{\pi}{3} + \cos\tfrac{\pi}{4}\cos\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}\cdot\tfrac{\sqrt{2}}{2}\cdot\tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{2}}{2}\cdot\tfrac{1}{2} = \tfrac{5\sqrt{2}}{8}\,. \end{aligned}$$

Thus the required distance is $c = R\cos^{-1}\left(\frac{5\sqrt{2}}{8}\right)$. The Law of Sines, modified for a sphere of radius R, gives

$$\frac{\sin\alpha}{\sin(a/R)} = \frac{\sin\gamma}{\sin(c/R)}$$

so we have

$$\sin\alpha = \frac{\sin(a/R)\sin\gamma}{\sin(c/R)} \cong \frac{\sin\frac{\pi}{6}\sin\frac{\pi}{4}}{\sqrt{1 - \left(\frac{5\sqrt{2}}{8}\right)^2}} = \frac{\frac{1}{2}\cdot\frac{\sqrt{2}}{2}}{\frac{\sqrt{14}}{8}} = \frac{2}{\sqrt{7}}\,.$$

Thus the bearing is θ east of north, where

$$\theta = \tfrac{\pi}{2} - \alpha = \tfrac{\pi}{2} - \sin^{-1}\left(\tfrac{2}{\sqrt{7}}\right) = \cos^{-1}\left(\tfrac{2}{\sqrt{7}}\right)\,.$$

(b) Find the radius R of a sphere on which there is a regular (equilateral) triangle with sides of length 1 and angles equal to $\frac{2\pi}{5}$.

Solution: We begin by finding $\cos\frac{2\pi}{5}$. We note that the polynomial $f(x) = x^5 - 1$ factors over $\mathbf{C}$ as

$$x^5 - 1 = (x-1)\left(x - e^{i\,2\pi/5}\right)\left(x - e^{-i\,2\pi/5}\right)\left(x - e^{i\,4\pi/5}\right)\left(x - e^{-i\,4\pi/5}\right)$$

and hence over $\mathbf{R}$ as

$$\begin{aligned} x^5 - 1 &= (x-1)\left(x^2 - 2\,\mathrm{Re}(e^{i\,2\pi/5}) + |e^{i\,2\pi/5}|^2\right)\left(x^2 - 2\,\mathrm{Re}(e^{i\,4\pi/5}) + |e^{i\,4\pi/5}|^2\right) \\ &= (x-1)\left(x^2 - 2\cos(\tfrac{2\pi}{5}) + 1\right)\left(x^2 - 2\cos(\tfrac{4\pi}{5}) + 1\right)\,. \end{aligned}$$

Writing $a = 2\cos\frac{2\pi}{5}$ and $b = 2\cos\frac{4\pi}{5}$, we need

$$(x^2 - ax + 1)(x^2 - bx + 1) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1\,.$$

Equating the coefficient of x^3 gives $-a - b = 1$ (1), and equating the coefficient of x^2 gives $2 + ab = 1$ (2). From equation (1) we have $b = -(1 + a)$, and putting this into equation (2) gives $a(1 + a) = 1$, that is $a^2 + a - 1 = 0$, and so we have $a = \frac{-1\pm\sqrt{5}}{2}$. Since $a = 2\cos\frac{2\pi}{5} > 0$ we must have $a = \frac{-1+\sqrt{5}}{2}$ and hence

$$\cos\tfrac{2\pi}{5} = \tfrac{-1+\sqrt{5}}{4}\,.$$

The Second Law of Cosines, modified for a sphere of radius R, is

$$\cos\tfrac{a}{R} = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\,.$$

Applying this to the triangle with $\alpha = \beta = \gamma = \frac{2\pi}{5}$ and $a = b = c = 1$, gives

$$\begin{aligned} \cos\tfrac{1}{R} &= \frac{\cos\frac{2\pi}{5} + \cos^2\frac{2\pi}{5}}{\sin^2\frac{2\pi}{5}} = \frac{\cos\frac{2\pi}{5} + \cos^2\frac{2\pi}{5}}{1 - \cos^2\frac{2\pi}{5}} = \frac{\frac{-1+\sqrt{5}}{4} + \frac{6-2\sqrt{5}}{16}}{1 - \frac{6-2\sqrt{5}}{16}} \\ &= \tfrac{-2+2\sqrt{5}+3-\sqrt{5}}{8-3+\sqrt{5}} = \tfrac{1+\sqrt{5}}{5+\sqrt{5}} = \tfrac{1+\sqrt{5}}{5+\sqrt{5}}\cdot\tfrac{5-\sqrt{5}}{5-\sqrt{5}} = \tfrac{4\sqrt{5}}{20} = \tfrac{1}{\sqrt{5}} \end{aligned}.$$

Thus $R = \dfrac{1}{\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)}$.

5: Let $u_1, u_2, \cdots, u_{n-2} \in \mathbf{R}^n$ and let $A = \big(u_1, u_2, \cdots, u_{n-2}\big) \in M_{n\times(n-2)}(\mathbf{R})$. For $i < j$, let $A^{i,j}$ denote the $(n-2)\times(n-2)$ matrix obtained from A by removing the i^{th} and j^{th} rows. Note that $\{u_1, \cdots, u_{n-2}\}$ is linearly independent if and only if $A^{i,j}$ is invertible for some $i < j$. Find a formula for an $n \times n$ matrix B with the property that if $\{u_1, \cdots, u_{n-2}\}$ is linearly dependent then $B = 0$ and if $\{u_1, \cdots, u_{n-2}\}$ is linearly independent then for all $i < j$, if $A^{i,j}$ is invertible the the i^{th} and j^{th} columns of B form a basis for $(\text{Col}A)^\perp$.

Solution: Let

$$B_{k,l} = \begin{cases} (-1)^{k+l}\big|A^{k,l}\big| & \text{, if } k < l \\ 0 & \text{, if } k = l \\ (-1)^{k+l+1}\big|A^{k,l}\big| & \text{, if } k > l\,. \end{cases}$$

If $\{u_1, u_2, \cdots, u_l\}$ is linearly dependent then each $\big|A^{k,l}\big| = 0$ and so $B = 0$. Suppose that $\{u_1, \cdots, u_l\}$ is linearly independent. Note that $\text{Col}A$ is $(n-2)$-dimensional and $(\text{Col}A)^\perp$ is 2-dimensional. We claim that each column of B lies in $(\text{Col}A)^\perp = \text{Null}(A^t)$, or equivalently that $\text{Col}B \subseteq \text{Null}(A^t)$. Writing A^l for the $(n-1)\times(n-2)$ matrix obtained from A by removing the l^{th} row and ${u_k}^l$ for the vector in $\mathbf{R}^{n-1}$ obtained from u_k by removing the l^{th} row, the (k,l) entry of the matrix A^tB is given by

$$\big[A^tB\big]_{k,l} = (k^{\text{th}} \text{ row of } A^t) \cdot (l^{\text{th}} \text{ column of } B) = u_k \cdot \begin{pmatrix} (-1)^{k+l}\big|A^{1,l}\big| \\ \vdots \\ -\big|A^{l-1,l}\big| \\ 0 \\ \big|A^{l+1,l}\big| \\ \vdots \\ (-1)^{n+l}\big|A^{n,l}\big| \end{pmatrix} = \Big|\big(A^l, {u_k}^l\big)\Big| = 0\,.$$

Thus $\text{Col}B \subseteq (\text{Col}A)^\perp$, as claimed. Next we note that for $i < j$ with $A^{i,j}$ invertible so that $\big|A^{i,j}\big| \neq 0$, we have

$$\begin{pmatrix} B_{i,i} & B_{i,j} \\ B_{j,i} & B_{j,j} \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{i+j}\big|A^{i,j}\big| \\ (-1)^{i+j+1}\big|A^{i,j}\big| & 0 \end{pmatrix}$$

which is clearly invertible, and so the i^{th} and j^{th} columns of B are linearly independent. Since these two columns span a 2-dimensional subspace of $\text{Col}B$ which is a subspace of the 2-dimensional space $(\text{Col}A)^\perp$, the two columns form a basis for $(\text{Col}A)^\perp$.