MATH 245 Linear Algebra 2, Solutions to Assignment 3

1: Let $u_1=\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}$, $u_2=\begin{pmatrix}2\\1\\1\\0\end{pmatrix}$, $u_3=\begin{pmatrix}1\\-3\\2\\1\end{pmatrix}$ and $x=\begin{pmatrix}1\\1\\7\\3\end{pmatrix}$. Let $\mathcal{U}=\{u_1,u_2,u_3\}$ and let $U=\text{Span}\,\mathcal{U}$. Find $\text{Proj}_U(x)$ in the following three ways.

(a) Let $A=\big(u_1,u_2,u_3\big)\in M_{4\times 3}$ then use the formula $\text{Proj}_U(x)=Ay$ where y is the solution to $A^tA\,y=A^tx$.

Solution: We have

$$A^tA=\begin{pmatrix}1&0&1&-1\\2&1&1&0\\1&-3&2&1\end{pmatrix}\begin{pmatrix}1&2&1\\0&1&-3\\1&1&2\\-1&0&1\end{pmatrix}=\begin{pmatrix}3&3&2\\3&6&1\\2&1&15\end{pmatrix}$$

$$A^tx=\begin{pmatrix}1&0&1&-1\\2&1&1&0\\1&-3&2&1\end{pmatrix}\begin{pmatrix}1\\1\\7\\3\end{pmatrix}=\begin{pmatrix}5\\10\\15\end{pmatrix}$$

$$\big(A^tA\big|A^tx\big)=\left(\begin{array}{ccc|c}3&3&2&5\\3&6&1&10\\2&1&15&15\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\3&6&1&10\\2&1&15&15\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\0&0&40&40\\0&3&-41&-35\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\0&3&-41&-35\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&0&3\\0&3&0&6\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&0&3\\0&1&0&2\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&-1\\0&1&0&2\\0&0&1&1\end{array}\right)$$

and so

$$y=\begin{pmatrix}-1\\2\\1\end{pmatrix}\ \text{ and }\ \text{Proj}_U(x)=Ay=\begin{pmatrix}1&2&1\\0&1&-3\\1&1&2\\-1&0&1\end{pmatrix}\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}4\\-1\\3\\2\end{pmatrix}.$$

(b) Apply the Gram-Schmidt Procedure to the basis $\mathcal{U}$ to obtain an orthogonal basis $\mathcal{V}=\{v_1,v_2,v_3\}$ for U, then use the formula $\text{Proj}_U(x)=\sum_{i=1}^{3}\frac{x\cdot v_i}{|v_i|^2}\,v_i$.

Solution: We let

$$v_1=u_1=\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}$$

$$v_2=u_2-\frac{u_2\cdot v_1}{|v_1|^2}\,v_1=\begin{pmatrix}2\\1\\1\\0\end{pmatrix}-\frac{3}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$$

$$v_3=u_3-\frac{u_3\cdot v_1}{|v_1|^2}\,v_1-\frac{u_3\cdot v_2}{|v_2|^2}\,v_2=\begin{pmatrix}1\\-3\\2\\1\end{pmatrix}-\frac{2}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2\\-8\\4\\6\end{pmatrix}=\frac{2}{3}\begin{pmatrix}1\\-4\\2\\3\end{pmatrix}.$$

Then

$$\text{Proj}_U(x)=\frac{x\cdot v_1}{|v_1|^2}\,v_1-\frac{x\cdot v_2}{|v_2|^2}\,v_2-\frac{x\cdot v_3}{|v_3|^2}\,v_3=\frac{5}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}+\frac{5}{3}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}+\frac{20}{30}\begin{pmatrix}1\\-4\\2\\3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}12\\-3\\9\\6\end{pmatrix}=\begin{pmatrix}1\\-1\\3\\2\end{pmatrix}.$$

(c) Find $w = X(u_1, u_2, u_3)$ so that $\{w\}$ is a basis for $U^\perp$, then calculate $\text{Proj}_U(x) = x - \text{Proj}_w(x)$.

Solution: We let

$$w = X(u_1,u_2,u_3) = X\left(\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-3\\2\\1\end{pmatrix}\right)$$

$$= \left(-\begin{vmatrix}0&1&-3\\1&1&2\\-1&0&1\end{vmatrix}, \begin{vmatrix}1&2&1\\1&1&2\\-1&0&1\end{vmatrix}, -\begin{vmatrix}1&2&1\\0&1&-3\\-1&0&1\end{vmatrix}, \begin{vmatrix}1&2&1\\0&1&-3\\1&1&2\end{vmatrix}\right)^t$$

$$= \begin{pmatrix}-(-2-1-3)\\(1-4-2+1)\\-(1+6+1)\\(2-6+3-1)\end{pmatrix} = \begin{pmatrix}6\\-4\\-8\\-2\end{pmatrix} = 2\begin{pmatrix}3\\-2\\-4\\-1\end{pmatrix}$$

and so

$$\text{Proj}_U(x) = x - \frac{x \bullet w}{|w|^2}\, w = \begin{pmatrix}1\\1\\7\\3\end{pmatrix} + \frac{30}{30}\begin{pmatrix}3\\-2\\-4\\-1\end{pmatrix} = \begin{pmatrix}4\\-1\\3\\2\end{pmatrix}.$$

2: Consider the vector space $P_2 = P_2(\mathbf{R})$ as a subspace of the vector space

$$\mathcal{C}\big((0,1)\big) = \mathcal{C}\big((0,1),\mathbf{R}\big) = \left\{ f : (0,1) \to \mathbf{R} \,\middle|\, f \text{ is continuous, and } \int_0^1 f(x)^2\,dx \text{ converges.}\right\}$$

with the inner product given by $\langle f, g\rangle = \int_0^1 fg$.

(a) Let $p_0 = 1$, $p_1 = x$ and $p_2 = x^2$. Apply the Gram-Schmidt Procedure to the basis $\{p_0, p_1, p_2\}$ to obtain an orthogonal basis $\{q_0, q_1, q_2\}$ for P_2.

Solution: We let

$$q_0 = p_0 = 1$$

$$q_1 = p_1 - \frac{\langle p_1, q_0\rangle}{|q_0|^2}\, q_0 = x - \tfrac{1/2}{1}\cdot 1 = x - \tfrac{1}{2}\ , \text{ and}$$

$$q_2 = p_2 - \frac{\langle p_2, q_0\rangle}{|q_0|^2}\, q_0 - \frac{\langle p_2, q_1\rangle}{|q_1|^2}\, q_1 = x^2 - \tfrac{1/3}{1}\cdot 1 - \tfrac{1/12}{1/12}\left(x - \tfrac{1}{2}\right) = x^2 - \tfrac{1}{3} - \left(x - \tfrac{1}{2}\right) = x^2 - x + \tfrac{1}{6}\,,$$

where we made use of the following equalities

$$\langle p_1, q_0\rangle = \int_0^1 x\,dx = \tfrac{1}{2}$$

$$|q_0|^2 = \int_0^1 1\,dx = 1$$

$$\langle p_2, q_0\rangle = \int_0^1 x^2\,dx = \tfrac{1}{3}$$

$$\langle p_2, q_1\rangle = \int_0^1 x^2\left(x - \tfrac{1}{2}\right)\,dx = \int_0^1 x^3 - \tfrac{1}{2}x^2\,dx = \tfrac{1}{4} - \tfrac{1}{6} = \tfrac{1}{12}$$

$$|q_1|^2 = \int_0^1 \left(x - \tfrac{1}{2}\right)^2\,dx = \int_0^1 x^2 - x + \tfrac{1}{4}\,dx = \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{1}{12}\,.$$

We also note (for use in parts (b) and (c)) that

$$|q_2|^2 = \int_0^1 \left(x^2 - x + \tfrac{1}{6}\right)^2 = \int_0^1 x^4 - 2x^3 + \tfrac{4}{3}x^2 - \tfrac{1}{3}x + \tfrac{1}{36}\,dx = \tfrac{1}{5} - \tfrac{1}{2} + \tfrac{4}{9} - \tfrac{1}{6} + \tfrac{1}{36} = \tfrac{1}{180}\,.$$

(b) Find the quadratic $f \in P_2$ which minimizes $\displaystyle\int_0^1 \left(f(x) - x^{-1/3}\right)^2 dx$.

Solution: Write $g(x) = x^{-1/3}$. Note that $\displaystyle\int_0^1 \left(f(x) - x^{-1/3}\right)^2 dx = \left|f - g\right|^2$. To minimize $|f - g|$ we must choose

$$\begin{aligned} f = \text{Proj}_{P_2}(g) &= \frac{\langle g, q_0\rangle}{|q_0|^2}\, q_0 + \frac{\langle g, q_1\rangle}{|q_1|^2}\, q_1 + \frac{\langle g, q_2\rangle}{|q_2|^2}\, q_2 \\ &= \tfrac{3/2}{1}\cdot 1 + \tfrac{-3/20}{1/12}\left(x - \tfrac{1}{2}\right) + \tfrac{1/40}{1/180}\left(x^2 - x + \tfrac{1}{6}\right) \\ &= \tfrac{3}{2} - \tfrac{9}{5}\left(x - \tfrac{1}{2}\right) + \tfrac{9}{2}\left(x^2 - x + \tfrac{1}{6}\right) = \tfrac{9}{2}x^2 - \tfrac{63}{10}x + \tfrac{63}{20}\,, \end{aligned}$$

where me made use of some equalities from part (a) along with the following

$$\begin{aligned} \langle g, q_0\rangle &= \int_0^1 x^{-1/3}\, dx = \tfrac{3}{2} \\ \langle g, q_1\rangle &= \int_0^1 \left(x - \tfrac{1}{2}\right) x^{-1/3}\, dx = \int_0^1 x^{2/3} - \tfrac{1}{2}x^{-1/3}\ dx = \tfrac{3}{5} - \tfrac{3}{4} = -\tfrac{3}{20} \\ \langle g, q_2\rangle &= \int_0^1 \left(x^2 - x + \tfrac{1}{6}\right) x^{-1/3}\, dx = \int_0^1 x^{5/3} - x^{2/3} + \tfrac{1}{6}x^{-1/3}\ dx = \tfrac{3}{8} - \tfrac{3}{5} + \tfrac{1}{4} = \tfrac{1}{40}\,. \end{aligned}$$

(c) Given that $f \in \mathcal{C}\big((0,1)\big)$ with $\displaystyle\int_0^1 f(x)\, dx = 3$, $\displaystyle\int_0^1 x f(x)\, dx = 2$ and $\displaystyle\int_0^1 x^2 f(x)\, dx = 1$, find the minimum possible value for $\displaystyle\int_0^1 f(x)^2\, dx$.

Solution: Since $\langle f, 1\rangle = 3$, $\langle f, x\rangle = 2$ and $\langle f, x^2\rangle = 1$ we have

$$\begin{aligned} &\langle f, q_0\rangle = \langle f, 1\rangle = 3 \\ &\langle f, q_1\rangle = \left\langle f, x - \tfrac{1}{2}\right\rangle = \langle f, x\rangle - \tfrac{1}{2}\langle f, 1\rangle = 2 - \tfrac{3}{2} = \tfrac{1}{2}\ \text{, and} \\ &\langle f, q_2\rangle = \left\langle f, x^2 - x + \tfrac{1}{6}\right\} = \langle f, x^2\rangle - \langle f, x\rangle + \tfrac{1}{6}\langle f, 1\rangle = 1 - 2 + \tfrac{1}{2} = -\tfrac{1}{2}\,, \end{aligned}$$

and hence

$$\text{Proj}_{P_2}(f) = \frac{\langle f, q_0\rangle}{|q_0|^2}\, q_0 + \frac{\langle f, q_1\rangle}{|q_1|^2}\, q_1 + \frac{\langle f, q_2\rangle}{|q_2|^2}\, q_2 = \tfrac{3}{1}\, q_0 + \tfrac{1/2}{1/12} q_1 + \tfrac{-1/2}{180} q_2 = 3q_0 + 6q_1 - 90q_2\,.$$

Let $g = 3q_0 + 6q_1 - 90q_2$. Given that $f \in \mathcal{C}(0,1)$ with $\text{Proj}_{P_2}(f) = g$, in order to minimize $|f|^2 = \displaystyle\int_0^1 f(x)^2\, dx$ we must choose $f = g$ since, by Pythagoras' Theorem, we have $|f|^2 = |g|^2 + |f - g|^2 \geq |g|^2$ with equality only when $|f - g| = 0$. Thus the minimum possible value for $|f|^2$ is

$$|g|^2 = \left\langle 3q_0 + 6q_1 - 90q_2, 3q_0 + 6q_1 - 90q_2\right\rangle = 9|q_0|^2 + 36|q_1|^2 + 8100|q_2|^2 = 9 + \tfrac{36}{12} + \tfrac{8100}{180} = 9 + 3 + 45 = 57\,.$$

3: Let U and V be inner product spaces over $\mathbf{R}$. An **isometry** from U to V is a surjective map $F : U \to V$ which preserves distance, so that for all $x, y \in U$ we have $\big|F(x) - F(y)\big| = |x - y|$. An inner product space **isomorphism** from U to V is a bijective linear map $G : U \to V$ which preserves inner product, so that for all $x, y \in U$ we have $\big\langle G(x), G(y)\big\rangle = \langle x, y\rangle$. Show that, in the case that U and V are finite dimensional, every isometry $F : U \to V$ is of the form $F(x) = G(x) + b$ for some inner product space isomorphism G and some $b \in V$.

Solution: Let $F : U \to V$ be an isometry. Note that F is bijective since it is surjective by definition and it is injective since, for $x, y \in U$ we have

$$F(x) = F(y) \Longrightarrow \big|F(x) - F(y)\big| = 0 \Longrightarrow |x - y| = 0 \Longrightarrow x = y\,.$$

Define $G : U \to V$ by

$$G(x) = F(x) - F(0)$$

so that we have $F(x) = G(x) + b$ with $b = F(0)$. Note that G is bijective, $G(0) = 0$ and G preserves distance since for $x, y \in U$ we have

$$\big|G(x) - G(y)\big| = \big|F(x) - F(0) - F(y) + F(0)\big| = \big|F(x) - F(y)\big| = |x - y|\,.$$

It follows from the Polarization Identity that G preserves inner product, indeed for $x, y \in U$ we have

$$\begin{aligned}
\big\langle G(x), G(y)\big\rangle &= \tfrac{1}{2}\big(|G(x)|^2 + |G(y)|^2 - |G(x) - G(y)|^2\big)\ \text{, by the Polarization Identity} \\
&= \tfrac{1}{2}\big(|G(x) - G(0)|^2 + |G(y) - G(0)|^2 - |G(x) - G(y)|^2\big)\ \text{, since } G(0) = 0 \\
&= \tfrac{1}{2}\big(|x - 0|^2 + |y - 0|^2 - |x - y|^2\big)\ \text{, since } G \text{ preserves distance} \\
&= \tfrac{1}{2}\big(|x|^2 + |y|^2 - |x - y|^2\big) \\
&= \langle x, y\rangle\ \text{, by the Polarization Identity.}
\end{aligned}$$

Finally, we provide two proofs that G is linear. For the first proof (which is valid even when U and V are infinite-dimensional), let $x, y \in U$ and $t \in \mathbf{R}$. Then we have $G(x + ty) = G(x) + tG(y)$ since

$$\begin{aligned}
\big|G(x+ty) - \big(G(x) + tG(y)\big)\big|^2 &= \big|G(x+ty) - G(x) - tG(y)\big|^2 \\
&= \big\langle G(x+ty) - G(x) - tG(y)\,,\, G(x+ty) - G(x) - tG(y)\big\rangle \\
&= \big\langle G(x+ty), G(x+ty)\big\rangle - \big\langle G(x+ty), G(x)\big\rangle - t\big\langle G(x+ty), G(y)\big\rangle \\
&\quad - \big\langle G(x), G(x+ty)\big\rangle + \big\langle G(x), G(x)\big\rangle + t\big\langle G(x), G(y)\big\rangle \\
&\quad - t\big\langle G(y), G(x+ty)\big\rangle + t\big\langle G(y), G(x)\big\rangle + t^2\big\langle G(y), G(y)\big\rangle \\
&= \langle x+ty, x+ty\rangle - \langle x+ty, x\rangle - t\langle x+ty, y\rangle \\
&\quad - \langle x, x+ty\rangle + \langle x, x\rangle + t\langle x, y\rangle \\
&\quad - \langle y, x+ty\rangle + t\langle y, x\rangle + t^2\langle y, y\rangle \\
&= \big\langle x + ty - x - ty\,,\, x + ty - x - ty\big\rangle \\
&= 0\,.
\end{aligned}$$

For the second proof, we begin by noting that G^{-1} also preserves inner product. Indeed, given $u, v \in V$, by writing $x = G^{-1}(u)$ and $y = G^{-1}(v)$ and using the fact that G preserves inner product, we have

$$\big\langle G^{-1}(u), G^{-1}(v)\big\rangle = \langle x, y\rangle = \big\langle G(x), G(y)\big\rangle = \langle u, v\rangle\,.$$

Next we show that $\dim(U) = \dim(V)$. Let $\mathcal{U} = \{u_1, \cdots, u_l\}$ be an orthonormal basis for U, and let $\mathcal{V} = \big\{G(u_1), \cdots, G(u_l)\big\}$. Since G preserves inner product, $\mathcal{V}$ is an orthonormal set, hence $\mathcal{V}$ is linearly independent, hence $\dim U = l \le \dim V$. Similarly, since G^{-1} also preserves inner product, $\dim V \le \dim U$. Thus $\dim U = \dim V$ and $\mathcal{V}$ is an orthonormal basis for V. Finally we note that G is linear since for $x = \sum_{i=1}^{l} t_i u_i \in U$ we can write $G(x) = \sum_{i=1}^{l} s_i G(u_i)$ for some $s_i \in \mathbf{R}$, and then for each i we have

$$\begin{aligned}
s_i &= \big\langle G(x), G(u_i)\big\rangle\ \text{, since } \mathcal{V} \text{ is orthonormal} \\
&= \langle x, u_i\rangle\ \text{, since } G \text{ preserves inner product} \\
&= t_i\ \text{, since } \mathcal{U} \text{ is orthonormal.}
\end{aligned}$$

4: Identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ using the bijection $\phi : \mathbf{C}^n \to \mathbf{R}^{2n}$ given by

$$\phi\big(x_1 + i\,y_1, \cdots, x_n + i\,y_n\big)^t = \big(x_1, y_1, \cdots, x_n, y_n\big)^t \,.$$

(a) Determine whether, for all vectors $u, v \in \mathbf{C}^n$, u is orthogonal to v in $\mathbf{C}^n$ if and only if $\phi(u)$ is orthogonal to $\phi(v)$ in $\mathbf{R}^{2n}$.

Solution: Note first that for $u, v \in \mathbf{C}^n$ given by $u = \begin{pmatrix} a_1 + i\,b_1 \\ \vdots \\ a_n + i\,b_n \end{pmatrix}$ and $v = \begin{pmatrix} c_1 + i\,d_1 \\ \vdots \\ c_n + i\,d_n \end{pmatrix}$, we have

$$\langle u, v\rangle = \sum_{k=1}^{n} (a_k + i\,b_k)(c_k - i\,d_k) = \sum_{k=1}^{n} (a_k c_k + b_k d_k) + i \sum_{k=1}^{n} (-a_k d_k + b_k c_k) \text{ , and}$$

$$\phi(u) \cdot \phi(v) = \sum_{k=1}^{n} (a_k c_k + b_k d_k) = \mathrm{Re}\big(\langle u, v\rangle\big) \,.$$

It follows that if $\langle u, v\rangle = 0$ then $\phi(u) \cdot \phi(v) = 0$, but the converse does not hold. For example for $u = e_1$ and $v = i\,e_1$ (where e_1 is the first standard basis vector) we have $\langle u, v\rangle = i$ and $\phi(u) \cdot \phi(v) = 0$.

(b) Determine whether, for all complex subspaces $U, V \subset \mathbf{C}^n$, U is orthogonal to V in $\mathbf{C}^n$ if and only if $\phi(U)$ is orthogonal to $\phi(V)$ in $\mathbf{R}^{2n}$.

Solution: This is true. Suppose first that U is orthogonal to V in $\mathbf{C}^n$ (so $\langle u, v\rangle = 0$ for all $u \in U, v \in V$). Given $x \in \phi(U)$ and $y \in \phi(V)$, let $u = \phi^{-1}(x)$ and $v = \phi^{-1}(y)$. Then (by our work in part (a))

$$x \cdot y = \phi(u) \cdot \phi(v) = \mathrm{Re}\big(\langle u, v\rangle\big) = \mathrm{Re}(0) = 0 \,.$$

Thus $\phi(U)$ is orthogonal to $\phi(V)$ in $\mathbf{R}^{2n}$. Conversely, suppose that $\phi(U)$ is orthogonal to $\phi(V)$ in $\mathbf{R}^{2n}$. Let $u \in U$ and $v \in V$. Note that we also have $iv \in V$. Then

$$0 = \phi(u) \cdot \phi(v) = \mathrm{Re}\big(\langle u, v\rangle\big) \text{ , and}$$
$$0 = \phi(u) \cdot \phi(iv) = \mathrm{Re}\big(\langle u, iv\rangle\big) = \mathrm{Re}\big(-i\langle u, v\rangle\big) = \mathrm{Im}\big(\langle u, v\rangle\big) \,.$$

Since $\mathrm{Re}\big(\langle u, v\rangle\big) = 0 = \mathrm{Im}\big(\langle u, v\rangle\big)$, it follows that $\langle u, v\rangle = 0$. Thus U is orthogonal to V in $\mathbf{C}^n$.

5: Identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ using the map ϕ from question 4. Given two 1-dimensional complex subspaces $U,V\subset\mathbf{C}^n$, we define the **angle** between U and V to be

$$\text{angle}(U,V)=\cos^{-1}\frac{|\langle u,v\rangle|}{|u||v|}\quad,\quad\text{where } 0\neq u\in U\ ,\ 0\neq v\in V\,.$$

(a) Explain why this definition is well-defined.

Solution: The definition makes sense firstly because $0\leq\dfrac{|\langle u,v\rangle|}{|u||v|}\leq 1$ by Cauchy's Inequality and secondly because the definition does not depend on the choice of u and v; indeed given $0\neq u'\in U$ and $0\neq v'\in V$ we have $u'=su$ and $v'=tv$ for some $0\neq s,t\in\mathbf{C}$ and so

$$\frac{|\langle u',v'\rangle|}{|u'||v'|}=\frac{\langle su,tv\rangle|}{|su||tv|}=\frac{|s\bar{t}\langle u,v\rangle|}{|s||u|\,|t||v|}=\frac{|\langle u,v\rangle|}{|u||v|}\,.$$

(b) Determine whether, for all 1-dimensional complex subspaces $U,V\subset\mathbf{C}^n$, the angle between U and V in $\mathbf{C}^n$ is equal to the angle between $\phi(U)$ and $\phi(V)$ in $\mathbf{R}^{2n}$.

Solution: This is true. When $U=V$ we have $\phi(U)=\phi(V)$ and $\text{angle}(U,V)=0=\text{angle}\big(\phi(U),\phi(V)\big)$. Suppose that $U\neq V$. Note that this implies that $U\cap V=\{0\}$ (since U and V are 1-dimensional). Let $X=\phi(U)$ and $Y=\phi(V)$. Note that X and Y are both 2-dimensional with $X\cap Y=\phi(U)\cap\phi(V)=\phi(U\cap V)=\{0\}$. For $0\neq u,v\in\mathbf{C}^n$ write

$$A(u,v)=\text{angle}\big(\text{Span}_{\mathbf{C}}\{u\}\,,\,\text{Span}_{\mathbf{C}}\{v\}\big)=\cos^{-1}\frac{|\langle u,v\rangle|}{|u||v|}$$

and for $0\neq x,y\in\mathbf{R}^{2n}$ write

$$B(x,y)=\text{angle}\big(\text{Span}_{\mathbf{R}}\{x\}\,,\,\text{Span}_{\mathbf{R}}\{y\}\big)=\cos^{-1}\frac{|x\bullet y|}{|x||y|}=\begin{cases}\theta(x,y) & \text{if } 0\leq\theta(x,y)\leq\frac{\pi}{2}\\ \pi-\theta(x,y) & \text{if } \frac{\pi}{2}\leq\theta(x,y)\leq\pi\,.\end{cases}$$

With this notation we have

$$\text{angle}(U,V)=A(u,v)\text{ where } 0\neq u\in U\,,\,0\neq v\in V$$
$$\text{angle}(X,Y)=\min\big\{B(x,y)\big|0\neq x\in X\,,\,0\neq y\in Y\big\}\,.$$

Recall that by our work in 4(a), when $x=\phi(u)$ and $y=\phi(v)$ we have $x\bullet y=\text{Re}\big(\langle u,v\rangle\big)$. Also note that $|x|^2=\langle x,x\rangle=\text{Re}\big(\langle u,u\rangle\big)=\text{Re}\big(|u|^2\big)=|u|^2$ so that $|x|=|u|$, and similarly $|y|=|v|$.

Given $0\neq x\in X$ and $0\neq y\in Y$, let $u=\phi^{-1}(x)$ and $v=\phi^{-1}(y)$. Note that $0\neq u\in U$ and $0\neq v\in V$ and we have $|x\bullet y|=\big|\text{Re}\big(\langle u,v\rangle\big)\big|\leq|\langle u,v\rangle|$ so that $\frac{|x\bullet y|}{|x||y|}\leq\frac{|\langle u,v\rangle|}{|u||v|}$ and hence $B(x,y)\geq A(u,v)$. It follows that $\text{angle}(U,V)\leq\text{angle}(X,Y)$.

Conversely, given $0\neq u\in U$ and $0\neq v\in V$, write $\langle u,v\rangle=r\,e^{i\,\theta}$ with $r=|\langle u,v\rangle|$. Note that $0\neq e^{i\,\theta}v\in V$ and $|e^{i\,\theta}v|=|v|$. Let $x=\phi(u)$ and $y=\phi(e^{i\,\theta}v)$ and note that $0\neq x\in X$ and $0\neq y\in Y$. We have

$$x\bullet y=\text{Re}\big(\langle u,e^{i\,\theta}v\rangle\big)=\text{Re}\big(e^{-i\,\theta}\langle u,v\rangle\big)=\text{Re}\big(e^{-i\,\theta}\,re^{i\,\theta}\big)=\text{Re}(r)=r=|\langle u,v\rangle|$$

so that $\frac{|x\bullet y|}{|x||y|}=\frac{|\langle u,v\rangle|}{|u||v|}$ and hence $B(x,y)=A(u,v)$. It follows that $\text{angle}(X,Y)\leq\text{angle}(U,V)$.