MATH 245 Linear Algebra 2, Solutions to Assignment 4

1: For $0 \neq u \in \mathbf{R}^3$ and $\theta \in \mathbf{R}$, let $R_{u,\theta} : \mathbf{R}^3 \to \mathbf{R}^3$ denote the rotation about the vector u by the angle θ (where the direction of rotation is determined by the right-hand rule: the right thumb points in the direction of u and the fingers curl in the direction of rotation).

(a) Let $u = (1, 1, -1)^t$ and let $\theta = \frac{\pi}{3}$. Find $A = [R_{u,\theta}]$.

Solution: Let $v = (0, 1, 1)^t$ and $w = (1, 0, 1)^t$. Let $\mathcal{U} = \{u, v, w\}$ and let $B = [R_{u,\theta}]_{\mathcal{U}}$. Note that v and w are othogonal to u with $|v| = |w| = \sqrt{2}$ and $v \times w = u$, and we have $\theta(v, w) = \cos^{-1} \frac{v \bullet w}{|v||w|} = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}$. Thus $R_{u,\theta}(u) = u$, $R_{u,\theta}(v) = w$ and $R_{u,\theta}(w) = w - v$, and so

$$B = [R_{u,\theta}]_{\mathcal{U}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} .$$

We have $A = PBP^{-1}$ where P is the change of basis matrix $P = [I]_{\mathcal{S}}^{\mathcal{U}} = (u_1, u_2, u_3)$. We calculate P^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right)$$

Thus

$$\begin{aligned} A = PBP^{-1} &= \frac{1}{3} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ -1 & 2 & -2 \\ -2 & 1 & 2 \end{pmatrix} . \end{aligned}$$

(b) Let $B = \begin{pmatrix} 2 & 3 & -6 \\ -3 & 6 & 2 \\ 6 & 2 & 3 \end{pmatrix}$. Find $c > 0$, $0 \neq u \in \mathbf{R}^3$ and $0 \leq \theta \leq \pi$ such that $B = \big[c\, R_{u,\theta}\big]$.

Solution: First, let us find the eigenvalues of the rotation $R_{u,\theta}$, where $0 \neq u \in \mathbf{R}^3$ and $\theta \in \mathbf{R}$. Let $u_1 = \frac{u}{|u|}$ and extend $\{u_1\}$ to an orthonormal basis $\mathcal{U} = \big\{u_1, u_2, u_3\big\}$ for $\mathbf{R}^3$. Writing $R = R_{u,\theta}$, we have

$$\big[R\big]_{\mathcal{U}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and so

$$\begin{aligned} f_R(t) &= \begin{vmatrix} 1-t & 0 & 0 \\ 0 & \cos\theta - t & -\sin\theta \\ 0 & \sin\theta & \cos\theta - t \end{vmatrix} = (1-t)\big((\cos\theta - t)^2 + \sin^2 t\big) \\ &= (1-t)(\cos^2\theta - 2\cos\theta\, t + t^2 + \sin^2\theta) = -(t-1)(t^2 - 2\cos\theta\, t + 1) \\ &= -(t-1)\big(t - e^{i\,\theta}\big)\big(t - e^{-i\,\theta}\big)\,. \end{aligned}$$

Thus the eigenvalues of $R = R_{u,\theta}$ are $1, e^{\pm i\,\theta}$. It follows that for $c > 0$, the eigenvalues of cR are $c, c\,e^{\pm i\,\theta}$. Now let us find the eigenvalues of B. We have

$$\begin{aligned} f_B(t) = \big|B - tI\big| &= \begin{vmatrix} 2-t & 3 & -6 \\ -3 & 6-t & 2 \\ 6 & 2 & 3-t \end{vmatrix} \\ &= (2-t)(18 - 9t + t^2) + 36 + 36 - 4(2-t) + 9(3-t) + 36(6-t) \\ &= 36 - 36\,t + 11\,t^2 - t^3 + 72 - 8 + 4t + 27 - 9t + 216 - 36\,t \\ &= -(t^3 - 11\,t^2 + 77\,t - 343) = -(t-7)(t^2 - 4t + 49) \end{aligned}$$

so the eigenvalues of B are $\lambda = 7$ or $\lambda = \frac{4 \pm \sqrt{16 - 4\cdot 49}}{2} = 2 \pm \sqrt{-45} = 2 \pm 3\sqrt{5}\,i$. Thus in order to have $B = \big[c\, R_{u,\theta}\big]$ with $c > 0$ and $0 \leq \theta \leq \pi$, we must have $c = 7$ and $\theta = \cos^{-1}\frac{2}{7}$. To find the required vector u, we find a basis for the eigenspace E_7. We have

$$B - 7I = \begin{pmatrix} -5 & 3 & -6 \\ -3 & -1 & 2 \\ 6 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -10 \\ -3 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -10 \\ 0 & 14 & -28 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -10 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

so a basis for E_7 is given by $\{u\}$ where $u = \pm\big(0,2,1\big)^t$. We still need to take some care in our choice of the vector u. If we chose $u = \big(0,2,1\big)^t$, $v = \big(0,-1,2\big)^t$, $w = \big(\sqrt{5},0,0\big)^t$ so that $\mathcal{U} = \{u,v,w\}$ is a positively oriented orthogonal basis with $|u| = |v| = |w|$, then we would have

$$B(u,v,w) = \begin{pmatrix} 2 & 3 & -6 \\ -3 & 6 & 2 \\ 6 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & \sqrt{5} \\ 2 & -1 & 0 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -15 & 2\sqrt{5} \\ 14 & -2 & -3\sqrt{5} \\ 7 & 4 & 6\sqrt{5} \end{pmatrix} = (u,v,w) \begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 3\sqrt{5} \\ 0 & -3\sqrt{5} & 2 \end{pmatrix}$$

so that $B = \big[c\, R_{u,-\theta}\big]$, which is not quite what we need. Instead we must choose $u = \big(0,-2,-1\big)^t$ (or some positive multiple of that) in order to get $B = \big[c\, R_{u,\theta}\big]$.

2: (a) Let $A = \begin{pmatrix} 0 & & & \\ \vdots & & I & \\ 0 & & & \\ a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix} \in M_{n\times n}(\mathbf{C})$. Find $f_A(t)$ and find a basis for each eigenspace E_λ.

Solution: The characteristic polynomial is

$$f_A(t) = \begin{vmatrix} -t & 1 & & 0 & 0 \\ & -t & \ddots & & \vdots \\ & & \ddots & 1 & 0 \\ 0 & & & -t & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1}-t \end{vmatrix}.$$

Expand along the last row to get

$$f_A(t) = (-1)^{n+1}a_0 \begin{vmatrix} 1 & & & 0 \\ -t & 1 & & \\ & \ddots & \ddots & \\ 0 & & -t & 1 \end{vmatrix} + \cdots + (-1)^{n+k+1}a_k \begin{vmatrix} -t & 1 & & & & & & \\ & -t & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & -t & & & & \\ & & & & 1 & & & \\ & & & & -t & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -t & 1 \end{vmatrix} + \cdots$$

$$\cdots + (-1)^{n+n-1}a_{n-2} \begin{vmatrix} -t & 1 & & & \\ & -t & \ddots & & \\ & & \ddots & 1 & \\ & & & -t & 1 \\ & & & & 1 \end{vmatrix} + (-1)^{n+n}(a_{n-1}-t) \begin{vmatrix} -t & 1 & & \\ & -t & \ddots & \\ & & \ddots & 1 \\ & & & -t & 1 \end{vmatrix}$$

$$= (-1)^{n+1}a_0 + \cdots + (-1)^{n+k+1}a_k(-t)^k + \cdots (-1)^{n+n-1}a_{n-2}(-t)^{n-1} + (-1)^{n+n}(a_{n-1}-t)(-t)^n$$

$$= (-1)^{n+1}\big(a_0 + \cdots + a_{n-2}t^{n-2} + a_{n-1}t^{n-1} - t^n\big).$$

Let λ be an eienvalue of A, so $f_A(\lambda) = 0$. We have

$$A - \lambda I = \begin{pmatrix} -t & 1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & -t & 1 & 0 \\ & & & -t & 1 \\ a_0 & & \cdots & a_{n-2} & a_{n-1}-\lambda \end{pmatrix}$$

The first $n-1$ rows (or if you prefer, the last $n-1$ columns) are clearly linearly independent so we must have $\dim E_\lambda = 1$. Let $u = \big(1, \lambda, \lambda^2, \cdots, \lambda^{n-1}\big)^t$. Notice that $(A-\lambda I)u = \big(0, \cdots, 0, (-1)^{n+1}f_A(\lambda)\big)^t = 0$ and so $\{u\}$ is a basis for E_λ.

(b) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{pmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.

Solution: By part (a) we have

$$f_A(t) = (-1)^4(6 + 5t - 2t^2 - t^3) = -(t^4 + 2t^2 - 5t - 6) = -(t+1)(t^2 + t - 6) = -(t+1)(t-2)(t+3)$$

so the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = -3$ and we can take $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) = \mathrm{diag}(-1, 2, -3)$. Also by part (a), we have corresponding eigenvectors $u_1 = (1, -1, 1)^t$, $u_2 = (1, 2, 4)^t$ and $u_3 = (1, -3, 9)^t$ so we can take $P = \big(u_1, u_2, u_3\big) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -3 \\ 1 & 4 & 9 \end{pmatrix}$.

(c) Let $x_0 = 2$, $x_1 = 2$ and $x_2 = 1$, and for $n \geq 0$ let $x_{n+3} = 6x_n + 5x_{n+1} - 2x_{n+2}$. Use part (b) to find x_n.

Solution: Since $x_3 = 6x_0 + 5x_1 - 2x_2$ we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

Similarly we have

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = A^3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix},$$

and so on. Thus

$$\begin{pmatrix} x_n \\ x_{n+1} \\ x_{n+2} \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = (PDP^{-1})^n \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = PD^nP^{-1} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

We calculate P^{-1}.

$$(P|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 10 & -2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{3}{15} & \frac{4}{15} & \frac{1}{15} \\ 0 & 0 & 1 & -\frac{1}{5} & -\frac{1}{10} & \frac{1}{10} \end{array}\right)$$

Thus we have

$$\begin{pmatrix} x_n \\ x_{n+1} \\ x_{n+2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} (-1)^n & & \\ & 2^n & \\ & & (-3)^n \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} \\ \frac{3}{15} & \frac{4}{15} & \frac{1}{15} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$x_n = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & & \\ & 2^n & \\ & & (-3)^n \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2}(-1)^n \\ 2^n \\ -\frac{1}{2}(-3)^n \end{pmatrix} = \frac{3}{2}(-1)^n + 2^n - \frac{1}{2}(-3)^n.$$

3: Let $A \in M_{n\times n}(\mathbf{R})$. Suppose that A is diagonalizable over $\mathbf{C}$, so there exists a diagonal matrix $D \in M_{n\times n}(\mathbf{C})$ and an invertible matrix $Q \in M_{n\times n}(\mathbf{C})$ such that $Q^{-1}AQ = D$. Show that there exists an invertible matrix $P \in M_{n\times n}(\mathbf{R})$ such that $P^{-1}AP$ is in the block-diagonal form

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_k & & & & & \\ & & & a_1 & b_1 & & & \\ & & & -b_1 & a_1 & & & \\ & & & & & \ddots & & \\ & & & & & & a_l & b_l \\ & & & & & & -b_l & a_l \end{pmatrix}$$

where each 1×1 block corresponds to a real eigenvalue λ_j of A, and each 2×2 block corresponds to a pair of conjugate complex eigenvalues $a_j \pm i\, b_j$.

Solution: Since A is diagonalizable over $\mathbf{C}$, we can choose a basis of complex eigenvectors for $\mathbf{C}^n$, say

$$\{v_1, \cdots, v_k, w_1, \cdots, w_l, z_1, \cdots, z_m\}$$

where each v_i is a complex eigenvector for a real eigenvalue λ_i, each w_α is a complex eigenvector for a complex eigenvalue $a_\alpha + i\, b_\alpha$ with $b_\alpha > 0$, and each z_β is a complex eigenvector for a complex eigenvalue $a_\beta - i\, b_\beta$ with $b_\beta > 0$. We make several preliminary remarks.

First we remark that the eigenvectors v_i can be chosen to be real vectors. This is because for a real eigenvalue λ, when we use our standard procedure to find a basis for the eigenspace $E_\lambda = \text{Null}(A - \lambda I)$ by reducing the real matrix $A - \lambda I$, we obtain a basis of real vectors.

Next we remark that we must have $m = l$ because the characteristic polynomial f_A is a real polynomial, so each complex root $\mu = a + i\, b$ occurs with the same algebraic multiplicity as its conjugate $\overline{\mu} = a - i\, b$, and since A is diagonalizable, the geometric and algebraic multiplicities are equal, so $\dim E_\lambda = \dim E_{\overline{\lambda}}$.

Finally, we remark that we can choose to have $z_\alpha = \overline{w}_\alpha$ for each $\alpha = 1, \cdots, l$. To see this, suppose that $\{w_1, \cdots, w_r\}$ is a basis for the eigenspace of $\mu = a + i\, b$. We claim that $\{\overline{w}_1, \cdots, \overline{w}_r\}$ is a basis for the eigenspace of $\overline{\mu}$. Since A is real, we have $A\overline{w}_\alpha = \overline{Aw_\alpha} = \overline{\mu w_\alpha} = \overline{\mu}\, \overline{w}_\alpha$, so each $\overline{w}_\alpha$ does lie in the eigenspace $E_{\overline{\mu}}$. Since $\{w_1, \cdots, w_r\}$ is linearly independent, it follows immediately that $\{\overline{w}_1, \cdots, \overline{w}_r\}$ is linearly independent. Since $\dim E_\mu = \dim E_{\overline{\mu}}$, it follows that $\{\overline{w}_1, \cdots, \overline{w}_r\}$ is a basis for $E_{\overline{\mu}}$, as claimed.

From these remarks, it follows that we can choose a basis of complex eigenvectors

$$\mathcal{V} = \{v_1, \cdots, v_k, w_1, \cdots, w_l, \overline{w}_1, \cdots, \overline{w}_l\}$$

for $\mathbf{C}^n$ such that each v_i is a real eigenvector for a real eigenvalue λ_i and each w_α is a complex eigenvector for a complex eigenvalue $\mu_\alpha = a_\alpha + i\, b_\alpha$ with $b_\alpha > 0$.

For each $\alpha = 1, \cdots, l$, write $w_\alpha = x_\alpha + i\, y_\alpha$ with $x_\alpha, y_\alpha \in \mathbf{R}^n$, and let

$$\mathcal{U} = \{v_1, \cdots, v_k, x_1, y_1, \cdots, x_l, y_l\}\,.$$

We claim that $\mathcal{U}$ is linearly independent over $\mathbf{R}$. Since $w_\alpha = x_\alpha + i\, y_\alpha$ and $\overline{w}_\alpha = x_\alpha - i\, y_\alpha$, we have $\text{Span}_{\mathbf{C}}\mathcal{V} \subset \text{Span}_{\mathbf{C}}\mathcal{U}$, and since $x_\alpha = \frac{1}{2}\, w_\alpha + \frac{1}{2}\, \overline{w}_\alpha$ and $y_\alpha = -\frac{i}{2}\, w_\alpha + \frac{i}{2}\, \overline{w}_\alpha$, we have $\text{Span}_{\mathbf{C}}\mathcal{U} \subset \text{Span}_{\mathbf{C}}\mathcal{V}$. Thus $\text{Span}_{\mathbf{C}}\mathcal{U} = \text{Span}_{\mathbf{C}}\mathcal{V}$, so $\mathcal{U}$ is a basis for $\mathbf{C}^n$. Since $\mathcal{U}$ is linearly independent over $\mathbf{C}$, it is also linearly independent over $\mathbf{R}$.

Let P be the matrix whose columns are the vectors in $\mathcal{U}$. Let B be the block-diagonal matrix in the statement of the theorem. We claim that $AP = PB$ so that $P^{-1}AP = B$. Since $Av_i = \lambda_i v_i$ for $1 \le i \le k$, it follows that the first k columns of AP are equal to those of PB. For $1 \le \alpha \le l$, since $Aw_\alpha = \mu_\alpha w_\alpha$ we have

$$A(x_\alpha + i\, y_\alpha) = (a_\alpha + i\, b_\alpha)(x_\alpha + i\, y_\alpha) = (a_\alpha x_\alpha - b_\alpha y_\alpha) + i(a_\alpha y_\alpha + b_\alpha x_\alpha)\,.$$

Equating real and imaginary parts gives

$$Ax_\alpha = a_\alpha x_\alpha - b_\alpha y_\alpha\ ,\quad Ay_\alpha = b_\alpha x_\alpha + b_\alpha y_\alpha\,.$$

It follows that the remaining columns of AP are equal to those of PB.

4: (a) Let U and V be inner product spaces over $\mathbf{C}$. Let $L : U \to V$ be a linear map, and suppose that the adjoint $L^* : V \to U$ exists. Show that $\text{Null}(L^*L) = \text{Null}(L) = \text{Range}(\text{L}^*)^\perp$.

Solution: First we show that $\text{Null}(L^*L) = \text{Null}(L)$. Let $x \in U$. If $L(x) = 0$ then $L^*L(x) = 0$, so we have $\text{Null}(L) \subset \text{Null}(L^*L)$. Conversely, if $L^*L(x) = 0$ then we have

$$|L(x)|^2 = \langle L(x), L(x)\rangle = \langle x, L^*L(x)\rangle = \langle x, 0\rangle = 0$$

so that $L(x) = 0$, and hence $\text{Null}(L^*L) \subset \text{Null}(L)$.

Next we show that $\text{Null}(L) = \text{Range}(\text{L}^*)^\perp$. Indeed for $x \in U$ we have

$$\begin{aligned} x \in \text{Null}(L) &\iff L(x) = 0 \\ &\iff \langle L(x), y\rangle = 0 \text{ for all } y \in V \\ &\iff \langle x, L^*(y)\rangle = 0 \text{ for all } y \in V \\ &\iff \langle x, z\rangle = 0 \text{ for all } z \in \text{Range}(\text{L}^*) \\ &\iff x \in \text{Range}(\text{L}^*)^\perp . \end{aligned}$$

(b) Let U be an inner product space over $\mathbf{C}$. Let $L : U \to U$ be linear and suppose that L^* exists. Show that $L = L^* \iff \big\langle L(x), x\big\rangle \in \mathbf{R}$ for all $x \in U$.

Solution: One direction is fairly easy. Indeed if $L = L^*$ then for all $x \in U$ we have

$$\langle L(x), x\rangle = \langle x, L^*(x)\rangle = \langle x, L(x)\rangle = \overline{\langle L(x), x\rangle}$$

and hence $\langle L(x), x\rangle \in \mathbf{R}$.

The other direction is more difficult. We first prove the following lemma:

Lemma: If $\langle L(x), x\rangle = 0$ for all $x \in U$ then $L = 0$.

Proof: Suppose that $\langle L(x), x\rangle = 0$ for all $x \in U$. Let $x, y \in U$. Then

$$0 = \langle L(x+y), (x+y)\rangle = \langle L(x), x\rangle + \langle L(x), y\rangle + \langle L(y), x\rangle + \langle L(y), y\rangle .$$

Since $\langle L(x), x\rangle = 0$ and $\langle L(y), y\rangle = 0$, this gives

$$0 = \langle L(x), y\rangle + \langle L(y), x\rangle \qquad (1).$$

Also, we have

$$0 = \langle L(x+iy), (x+iy)\rangle = \langle L(x), x\rangle + \langle L(x), iy\rangle + \langle iL(y), x\rangle + \langle iL(y), iy\rangle = -i\langle L(x), y\rangle + i\langle L(y), x\rangle .$$

Multiplying both sides by i gives

$$0 = \langle L(x), y\rangle - \langle L(y), x\rangle \qquad (2).$$

Solving equations (1) and (2) gives $\langle L(x), y\rangle = 0$ and $\langle L(y), x\rangle = 0$. Since $\langle L(x), y\rangle = 0$ for all $y \in U$, we know that $L(x) = 0$. Since $L(x) = 0$ for all $x \in U$, we know that $L = 0$. This proves the lemma.

Now we use the lemma to prove that if $\langle L(x), x\rangle \in \mathbf{R}$ for all $x \in U$ then we must have $L = L^*$. Suppose that $\langle L(x), x\rangle \in \mathbf{R}$ for all $x \in U$. Let $x \in U$. Since $\langle L(x), x\rangle \in \mathbf{R}$, we have

$$\langle L(x), x\rangle = \overline{\langle L(x), x\rangle} = \overline{\langle x, L^*(x)\rangle} = \langle L^*(x), x\rangle$$

and so

$$0 = \langle L(x), x\rangle - \langle L^*(x), x\rangle = \big\langle (L - L^*)(x), x\big\rangle .$$

Since $\big\langle (L - L^*)(x), x\big\rangle = 0$ for all $x \in U$, it follows from the above lemma that $L - L^* = 0$, that is $L = L^*$.

5: Let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Let V be the inner product space over $\mathbf{F}$ consisting of all sequences $a = (a_1, a_2, a_3, \cdots)$ with each $a_k \in \mathbf{F}$ such that only finitely many of the terms a_k are non-zero, with the inner product given by $\langle a, b\rangle = \sum\limits_{k=1}^{\infty} a_k \overline{b_k}$. Let $U = \left\{ a = (a_1, a_2, \cdots) \in V \middle| \sum\limits_{k=1}^{\infty} a_k = 0 \right\}$. The standard basis for V is the basis $\mathcal{S} = \{e_1, e_2, e_3, \cdots\}$ where $e_n = (e_{n,1}, e_{n,2}, e_{n,3}, \cdots)$ with $e_{n,k} = \delta_{n,k}$.

(a) Show that $U^\perp = \{0\}$.

Solution: Let $a = (a_1, a_2, \cdots) \in U^\perp$. Since only finitely many of the terms a_k are non-zero, we can choose a positive integer n so that $a_k = 0$ for all $k > n$, that is $a = (a_1, a_2, \cdots, a_n, 0, 0, \cdots)$. For each $k = 1, 2, \cdots, n$ notice that $e_k - e_{n+1} \in U$, so since $a \in U^\perp$ we have

$$0 = \langle a, e_k - e_{n+1}\rangle = \langle a, e_k\rangle - \langle a, e_{n+1}\rangle = a_k - a_{n+1} = a_k\,.$$

(b) Show that $\dim(U^0) = 1$.

Solution: Define $f : V \to \mathbf{F}$ by $f(a) = \sum\limits_{k=1}^{\infty} a_k$. Note that f is well-defined since only finitely many of the terms a_k are non-zero, and f is linear, so we have $f \in V^*$. We claim that $U^0 = \text{Span}\{f\}$. For $a = (a_1, a_2, \cdots) \in U$, we have $f(a) = \sum\limits_{k=1}^{\infty} a_k = 0$, so $f \in U^0$ and hence $\text{Span}\{f\} \subset U^0$. Conversely, let $g \in U^0$ so that $g(u) = 0$ for all $u \in U$. Notice that for all $k = 1, 2, 3, \cdots$ we have $e_1 - e_k \in U$, so $0 = g(e_1 - e_k) = g(e_1) - g(e_k)$, and hence $g(e_k) = g(e_1)$. For all $a \in V$, we have

$$g(a) = g\big(\sum_{k=1}^{\infty} a_k e_k\big) = \sum_{k=1}^{\infty} a_k\, g(e_k) = \sum_{k=1}^{\infty} a_k\, g(e_1) = \big(\sum_{k=1}^{\infty} a_k\big)\, g(e_1) = g(e_1)\, f(a)\,.$$

Thus $g = g(e_1)\, f \in \text{Span}\{f\}$ and hence $U^0 \subset \text{Span}\{f\}$.

(c) Let $\mathcal{F} = \{f_1, f_2, f_3, \cdots\}$ where $f_n \in V^*$ is determined by $f_n(e_k) = \delta_{n,k}$. Show that $\mathcal{F}$ is linearly independent but does not span V^*.

Solution: We claim that $\mathcal{F}$ is linearly independent. Suppose that some (finite) linear combination of the elements of $\mathcal{F}$ is equal to zero, say $\sum\limits_{i=1}^{n} c_i\, f_i = 0$. Then for every $a \in V$ we have $\sum\limits_{i=1}^{n} c_i\, f_i(a) = 0$, and in particular for every $k = 1, 2, 3, \cdots$ we have $0 = \sum\limits_{i=1}^{n} c_i\, f_i(e_k) = \sum\limits_{i=1}^{n} c_i\, \delta_{i,k} = c_k$. Thus $\mathcal{F}$ is linearly independent.

On the other hand, we claim that $\mathcal{F}$ does not span V^*. Let $f \in V^*$ be the map from part (b) given by $f(a) = \sum\limits_{k=1}^{\infty} a_k$. Notice that f cannot be equal to any (finite) linear combination of the elements of $\mathcal{F}$, since for $g = \sum\limits_{i=1}^{n} c_i\, f_i$ we have $g(e_{n+1}) = 0$ while $f(e_{n+1}) = 1$, so $g \neq f$. Thus $\mathcal{F}$ does not span V^*.

(d) Define $E : V \to V^{**}$ by $E(a)(f) = f(a)$, where $a \in V$ and $f \in V^*$. Show that E is 1:1 but not onto.

Solution: Note that E is linear, so to show that E is 1:1 it suffices to show that $\text{Null}(E) = \{0\}$. Let $a \in \text{Null}(E)$ so $E(a) = 0$. Then for all $f \in V^*$ we have $f(a) = E(a)(f) = 0$. In particular, for all $k = 1, 2, 3 \cdots$ we have

$$0 = f_k(a) = f_k\big(\sum_{i=1}^{\infty} a_i\, e_i\big) = \sum_{i=1}^{\infty} f_k(e_i) = \sum_{i=1}^{\infty} a_i\, \delta_{k,i} = a_k$$

and so $a = 0$.

We claim that E is not onto. Extend the linearly independent set $\mathcal{F}$ to a basis $\mathcal{F} \cup \mathcal{G}$ for V^* (where $\mathcal{F}$ and $\mathcal{G}$ are disjoint). Let $h : V^* \to \mathbf{F}$ be the (unique) linear map given by $h(f_k) = 1$ for all $k = 1, 2, 3, \cdots$ and $h(g) = 0$ for all $g \in \mathcal{G}$. Notice that h cannot be in the range of E since given $a = (a_1, a_2, \cdots) \in V$ we can choose k so that $a_k = 0$, and then we have $E(a)(f_k) = f_k(a) = 0$ while $h(f_k) = 1$, so $E(a) \neq h$.

(e) Define $L : V \to V$ by $L(a)_k = \sum\limits_{i=k}^{\infty} a_i$, where $a \in V$. Show that L has no adjoint.

Solution: Notice that for all $k = 1, 2, 3, \cdots$ we have $L(e_k) = (1, 1, \cdots, 1, 0, 0, 0, \cdots) = \sum\limits_{i=1}^{k} e_i$, and so $\langle L(e_k), e_1\rangle = 1$. Suppose, for a contradiction, that L had an adjoint L^*. Let $a = L^*(e_1) \in V$. Choose k so that $a_k = 0$. Then $\langle e_k, a\rangle = \overline{a_k} = 0$. But this contradicts the fact that $\langle e_k, a\rangle = \langle e_k, L^*(e_1)\rangle = \langle L(e_k), e_1\rangle = 1$.