MATH 245 Linear Algebra 2, Solutions to Assignment 5

1: (a) Let $A=\begin{pmatrix}3&2&-4\\2&0&-2\\-4&-2&3\end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^tAP=D$.

Solution: The characteristic polynomial of A is

$$\begin{aligned}f_A(t)&=\begin{vmatrix}3-t&2&-4\\2&-t&-2\\-4&-2&3-t\end{vmatrix}\\&=-t(t^3-6t+9)+16+16-4(3-t)-4(3-t)+16t\\&=-(t^3-6t^2-15t-8)=-(t+1)(t^2-7t-8)=-(t+1)^2(t-8)\end{aligned}$$

so the eigenvalues are $\lambda_1=8$ and $\lambda_2=\lambda_3=-1$. When $\lambda=\lambda_1=8$ we have

$$A-\lambda I=\begin{pmatrix}-5&2&-4\\2&-8&-2\\-4&-2&-5\end{pmatrix}\sim\begin{pmatrix}1&-4&-1\\2&-8&-2\\-4&-2&-5\end{pmatrix}\sim\begin{pmatrix}1&-4&-1\\0&0&0\\0&-18&-9\end{pmatrix}\sim\begin{pmatrix}1&-4&-1\\0&1&\frac12\\0&0&0\end{pmatrix}\sim\begin{pmatrix}1&0&1\\0&1&\frac12\\0&0&0\end{pmatrix}$$

so we can take $v_1=\begin{pmatrix}2\\1\\-2\end{pmatrix}$ so that $\{v_1\}$ is a basis for the eigenspace E_8, then let $u_1=\dfrac{v_1}{|v_1|}=\frac13\begin{pmatrix}2\\1\\-2\end{pmatrix}$. Since A is symmetric, we know that its eigenspaces are orthogonal so we have $E_{-1}={E_8}^\perp$. To find a basis for E_{-1} we can, by inspection, choose a unit vector u_2 which is orthogonal to u_1 and then choose $u_3=u_1\times u_2$. We choose $u_2=\frac{1}{\sqrt2}\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $u_3=\frac{1}{3\sqrt2}\begin{pmatrix}2\\1\\-2\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix}=\frac{1}{3\sqrt2}\begin{pmatrix}1\\-4\\-1\end{pmatrix}$. Thus we can orthogonally diagonalize A using $P=\big(u_1,u_2,u_3\big)=\begin{pmatrix}\frac23&\frac{1}{\sqrt2}&\frac{1}{3\sqrt2}\\\frac13&0&-\frac{4}{3\sqrt2}\\-\frac23&\frac{1}{\sqrt2}&-\frac{1}{3\sqrt2}\end{pmatrix}$ and $D=\text{diag}(\lambda_1,\lambda_2,\lambda_3)=\begin{pmatrix}8&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$.

(b) Let $A = \begin{pmatrix} 2+i & 1+i \\ i & 3+i \end{pmatrix}$. Find a unitary matrix P and an upper-triangular matrix T so that $P^*AP = T$.

Solution: The characteristic polynomial of A is

$$\begin{aligned} f_A(t) &= \begin{vmatrix} (2+i)-t & 1+i \\ i & (3+i)-t \end{vmatrix} \\ &= t^2 - (5+2i)\,t + (5+5i) - (-1+i) = t^2 - (5+2i)\,t + (6+4i) \end{aligned}$$

The eigenvalues are

$$\lambda = \frac{(5+2i) \pm \sqrt{(21+20\,i) - (24+16\,i)}}{2} = \frac{(5+2i) \pm \sqrt{-3+4\,i}}{2} = \frac{(5+2i) \pm (1+2i)}{2} = 3+2i\,,\ 2,$$

say $\lambda_1 = 2$ and $\lambda_2 = 3+2i$. When $\lambda = \lambda_1 = 2$ we have

$$A - \lambda I = \begin{pmatrix} i & 1+i \\ i & 1+i \end{pmatrix} \sim \begin{pmatrix} 1 & 1-i \\ 0 & 0 \end{pmatrix}$$

so we can take $v_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$ so that $\{v_1\}$ is a basis for E_{λ_1}. By inspection, the vector $v_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ is orthogonal to v_1. Normalize these vectors to get $u_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$ and $u_2 = \frac{v_2}{|v_2|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ so that $\{u_1, u_2\}$ is an orthonormal basis for $\mathbf{C}^2$. Thus we can take

$$P = \big(u_1, u_2\big) = \tfrac{1}{\sqrt{3}} \begin{pmatrix} -1+i & 1 \\ 1 & 1+i \end{pmatrix}\ , \text{ and}$$

$$\begin{aligned} T = P^*AP &= \tfrac{1}{3} \begin{pmatrix} -1-i & 1 \\ 1 & 1-i \end{pmatrix} \begin{pmatrix} 2+i & 1+i \\ i & 3+i \end{pmatrix} \begin{pmatrix} -1+i & 1 \\ 1 & 1+i \end{pmatrix} \\ &= \tfrac{1}{3} \begin{pmatrix} -1-i & 1 \\ 1 & 1-i \end{pmatrix} \begin{pmatrix} -2+2i & 2+3i \\ 2 & 2+5i \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} 6 & 3 \\ 0 & 9+6i \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3+2i \end{pmatrix}\ . \end{aligned}$$

2: Find a singular value decomposition $A = Q\Sigma P^*$ for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \\ 1 & 1 \end{pmatrix}$.

Solution: We have

$$A^*A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ 6 & 6 \end{pmatrix}.$$

The characteristic polynomial of A^*A is

$$f_{A^*A}(t) = \left|A^*A - tI\right| = \begin{vmatrix} 15-t & 6 \\ 6 & 6-t \end{vmatrix} = t^2 - 21t + 54 = (t-3)(t-18)$$

so the eigenvalues of A^*A are $\lambda_1 = 18$, $\lambda_2 = 3$, and hence the singular values of A are $\sigma_1 = 3\sqrt{2}$, $\sigma_2 = \sqrt{3}$, so we can take

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

When $\lambda = \lambda_1 = 18$ we have

$$A^*A - \lambda I = \begin{pmatrix} -3 & 6 \\ 6 & -12 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$$

so $u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a unit eigenvector for λ_1. The eigenspace for λ_2 is orthogonal so $u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is a unit eigenvector for λ_2, and so we can take

$$P = (u_1, u_2) = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

Next let $v_1 = \frac{Au_1}{\sigma_1} = \frac{1}{3\sqrt{10}} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{3\sqrt{10}} \begin{pmatrix} 4 \\ 4 \\ 7 \\ 3 \end{pmatrix}$ and $v_2 = \frac{Au_2}{\sigma_2} = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{15}} \begin{pmatrix} 3 \\ -2 \\ -1 \\ 1 \end{pmatrix}$ then extend $\{v_1, v_2\}$ to an orthonormal basis $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ for $\mathbf{R}^4$. We have

$$\begin{pmatrix} 4 & 4 & 7 & 3 \\ 3 & -2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 8 & 2 \\ 3 & -2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 8 & 2 \\ 0 & 20 & 25 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 8 & 2 \\ 0 & 1 & \frac{5}{4} & \frac{1}{4} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} & \frac{1}{4} \end{pmatrix}$$

So the orthogonal complement of $\text{Span}\{v_1, v_2\}$ has basis $\left\{(2,1,0,-4)^t, (2,5,-4,0)^t\right\}$. We apply the Gram-Schmidt Procedure, replacing the second vector in the basis by

$$\begin{pmatrix} 2 \\ 5 \\ -4 \\ 0 \end{pmatrix} - \frac{9}{21}\begin{pmatrix} 2 \\ 1 \\ 0 \\ -4 \end{pmatrix} = \frac{1}{7}\left(\begin{pmatrix} 14 \\ 35 \\ -28 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \\ 0 \\ -12 \end{pmatrix}\right) = \frac{1}{7}\begin{pmatrix} 8 \\ 32 \\ -28 \\ 12 \end{pmatrix} = \frac{4}{7}\begin{pmatrix} 2 \\ 8 \\ -7 \\ 3 \end{pmatrix},$$

and then we normalize to obtain $v_3 = \frac{1}{\sqrt{21}}\begin{pmatrix} 2 \\ 1 \\ 0 \\ -4 \end{pmatrix}$ and $v_4 = \frac{1}{3\sqrt{14}}\begin{pmatrix} 2 \\ 8 \\ -7 \\ 3 \end{pmatrix}$. Thus we can take

$$Q = \left(v_1, v_2, v_3, v_4\right) = \begin{pmatrix} \frac{4}{3\sqrt{10}} & \frac{3}{\sqrt{15}} & \frac{2}{\sqrt{21}} & \frac{2}{3\sqrt{14}} \\ \frac{4}{3\sqrt{10}} & \frac{-2}{\sqrt{15}} & \frac{1}{\sqrt{21}} & \frac{8}{3\sqrt{14}} \\ \frac{7}{3\sqrt{10}} & \frac{-2}{\sqrt{15}} & \frac{0}{\sqrt{21}} & \frac{-7}{3\sqrt{14}} \\ \frac{3}{3\sqrt{10}} & \frac{1}{\sqrt{15}} & \frac{-4}{\sqrt{21}} & \frac{3}{3\sqrt{14}} \end{pmatrix}.$$

3: A matrix $A \in M_{n\times n}(\mathbf{C})$ is called Hermitian positive-definite when $A^* = A$ and the eigenvalues of A are all positive. Let H_n denote the set of Hermitian positive-definite matrices in $M_{n\times n}(\mathbf{C})$.

(a) Show that every element of H_n has a unique square root in H_n.

Solution: Let $A \in H_n$. Since $A = A^*$ we can unitarily diagonalize A. Choose a unitary matrix P so that $P^*AP = D = \text{diag}\{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ (we know the eigenvalues of A are positive since $A \in H_n$). Let $E = \text{diag}\big(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}\big)$ and note that $E^2 = D$. Let $B = PEP^*$. Note that $B \in H_n$ since $B^* = B$ and the eigenvalues of B are $\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}$ which are positive. Also note that $B^2 = (PEP^*)^2 = PEP^*PEP^* = PEP^* = A$. Thus B is a square root of A in H_n.

It remains to show that this square root is unique. Suppose that $C \in H_n$ and $C^2 = A$. Since $C^* = C$ we can unitarily diagonalize C. Choose a unitary matrix Q so that $Q^*CQ = F = \text{diag}\big(\sigma_1, \cdots, \sigma_n\big)$ with $\sigma_1 \geq \cdots \geq \sigma_n > 0$ (each $\sigma_i > 0$ since $C \in H_n$) and let $v_1, \cdots, v_n$ be the columns of Q. For each i we have

$$Au_i = C^2u_i = C(Cu_i) = C(\sigma_i u_i) = \sigma_i Cu_i = {\sigma_i}^2 u_i$$

so the eigenvalues of C are the square roots of the eigenvalues of A and each eigenvector of C is also an eigenvector for A. It follows that $\sigma_i = \sqrt{\lambda_i}$ for each i, and that $\{v_1, \cdots, v_n\}$ is both a basis of eigenvectors for C and a basis of eigenvectors for A, so the matrices C and A have the same eigenvectors. Similarly the above matrix B also has the same eigenvectors. It follows that the unitary matrix Q which we used to diagonalize C can also be used to diagonalize B. Thus we have

$$Q^*CQ = \text{diag}\big(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}\big) = Q^*BQ\,.$$

Multiplying on the left by Q and on the right by Q^* gives $C = B$.

(b) Let $A \in H_n$. Show that if $A = Q\Sigma P^*$ is a singular value decomposition of A, then $Q = P$.

Solution: Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A^*A, let $\sigma_i = \sqrt{\lambda_i}$ be the singular values of A, let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0$ be the eigenvalues of A, and let $\mathcal{U} = \{u_1, \cdots, u_n\}$ be a corresponding basis of eigenvectors of A. Since A is Hermitian (that is $A^* = A$) and since $Au_i = \mu_i u_i$, we have

$$A^*Au_i = AAu_i = A\mu_i u_i = \mu_i Au_i = {\mu_i}^2 u_i$$

and so each ${\mu_i}^2$ is an eigenvalue of A^*A with eigenvector u_i. Since A is positive semi-definite we must have $\mu_i = \sqrt{\lambda_i} = \sigma_i$, and so the eigenvalues of A are equal to the singular values of A, and A and A^*A have the same eigenvectors.

Let $A = Q\Sigma P^*$, with $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n) = \text{diag}(\mu_1, \cdots, \mu_n)$, be a singular value decomposition of A. Let $w_1, \cdots, w_n$ be the columns of P and let $\mathcal{W} = \{w_1, \cdots, w_n\}$. Recall that $\mathcal{W}$ is a basis of eigenvectors of A^*A. Since A and A^*A have the same eigenvectors, $\mathcal{W}$ is also a basis of eigenvectors of A, and so it diagonalizes A, that is $A = P\Sigma P^*$. Thus we have $A = Q\Sigma P^* = P\Sigma P^*$. Multiply on the right by P and then by $\Sigma^{-1} = \text{diag}\big(\frac{1}{\sigma_1}, \cdots, \frac{1}{\sigma_n}\big)$ to get $Q = P$.

4: Let $A \in M_{n\times n}(\mathbf{C})$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A (listed with repetition according to algebraic multiplicity). Show that the following are equivalent.

1. $A\,A^* = A^*A$.
2. $A^* = f(A)$ for some polynomial f.
3. $A^* = AP$ for some unitary matrix P.
4. $\sum_{i,j} \left|A_{i,j}\right|^2 = \sum_i |\lambda_i|^2$.

Solution: First we show that $1 \iff 2$. Suppose first that $A^*A = A\,A^*$. Choose a unitary matrix P so that $P^*AP = D = \text{diag}\big(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2, \cdots, \lambda_k, \cdots, \lambda_k\big)$ where $\lambda_1, \cdots, \lambda_k$ are the distinct eigenvalues of A. Note that $A = P\,DP^*$, $A^* = P\,D^*P^*$ and $D^* = \overline{D}$. Let f be the unique polynomial of degree at most $k-1$ such that $f(\lambda_i) = \overline{\lambda_i}$ for $i = 1, 2, \cdots, k$. Note that $f(D) = \overline{D}$, so we have

$$f(A) = f(PDP^*) = Pf(D)\,P^* = P\,\overline{D}\,P^* = PD^*P^* = A^*\,.$$

Conversely, for any square matrix A and any polynomial f, it is clear that A commutes with $f(A)$ $\big($indeed if $f(x) = \sum c_k x^k$ then we have $Af(A) = \sum c_k A^{k+1} = f(A)A$ $\big)$, so if $A^* = f(A)$ then A commutes with A^*.

Next we show that $1 \iff 3$. Suppose first that $A\,A^* = A^*A$. Choose a unitary matrix Q so that $Q^*AQ = D = \text{diag}(\lambda_1, \cdots, \lambda_n)$ where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A (listed with repetition according to multiplicity). Note that $A = QDQ^*$ and $A^* = QD^*Q^*$. Let $E = \text{diag}\big(\frac{\overline{\lambda_1}}{\lambda_1}, \cdots, \frac{\overline{\lambda_n}}{\lambda_n}\big)$. Note that E is unitary and $DE = \text{diag}(\overline{\lambda}_1, \cdots, \overline{\lambda}_n) = D^*$. Let $P = QEQ^*$. Then P is unitary and

$$AP = QD\,Q^*QEQ^* = QDEQ^* = QD^*Q^* = A^*\,.$$

Conversely, if $A^* = AP$ where P is unitary, then $A^*A = A^*A^{**} = (AP)(AP)^* = APP^*A^* = AA^*$.

Finally we show that $1 \iff 4$. Suppose first that $AA^* = A^*A$. Choose a unitary matrix P so that $P^*AP = D = \text{diag}(\lambda_1, \cdots, \lambda_n)$. Recall that similar matrices have the same trace (since they have the same eigenvalues and the trace of a square matrix is the sum of its eigenvalues). Note that A^*A is similar to D^*D since $D^*D = PA^*P^*PAP^* = PA^*AP^*$. Using the standard inner product $\langle A, B\rangle = \text{trace}(B^*A)$, we have

$$\sum_{i,j} |A_{i,j}|^2 = |A|^2 = \text{trace}(A^*A) = \text{trace}(D^*D) = |D|^2 = \sum_i |\lambda_i|^2\,.$$

Conversely, suppose that $A^*A \neq AA^*$. Choose a unitary matrix P so that $P^*AP = T$ is upper triangular with diagonal entries $T_{i,i} = \lambda_i$. Since $A^*A \neq AA^*$, we know that T is not diagonal, so we have $T_{k,l} \neq 0$ for some $k < l$. Then

$$\sum_{i,j} |A_{i,j}|^2 = |A|^2 = |T|^2 = \sum_{i\leq j} |T_{i,j}|^2 \geq |T_{k,l}|^2 + \sum_i |T_{i,i}|^2 = |T_{k,l}|^2 + \sum_i |\lambda_i|^2 > \sum_i |\lambda_i|^2\,.$$

5: Let $A \in O(3,\mathbf{R})$ and let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be the associated linear map given by $L(x) = Ax$. Show that L is either a rotation, a reflection in some 2-dimensional subspace of $\mathbf{R}^3$, or a rotary inversion (that is a map of the form $-R$ where R is a rotation).

Solution: Either the eigenvalues of A are all real or A has one real eigenvalue and a pair of conjugate eigenvalues. Suppose first that the eigenvalues of A are all real. Since $A^*A = A^tA = I$ and the eigenvalues of A are real, A is orthogonally diagonalizable over $\mathbf{R}$ and its eigenvalues all have length 1. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ with $\lambda_i = \pm 1$ be the eigenvalues of A. Let $\mathcal{U} = \{u_1, u_2, u_3\}$ be an orthonormal basis of corresponding eigenvectors and let $P = \big(u_1, u_2, u_2\big)$ so that we have

$$\big[L\big]_{\mathcal{U}} = P^tAP = \text{diag}\big(\lambda_1, \lambda_2, \lambda_3\big)\,.$$

When $\lambda_1 = \lambda_2 = \lambda_3 = 1$ we have $L = I = R_{u_1,0}$. When $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$ we have $L = \text{Refl}_{E_1}$ where $E_1 = \text{Span}\{u_1, u_2\}$. When $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$ we have $L = \mathbf{R}_{u_1,\pi}$. When $\lambda_1 = \lambda_2 = \lambda_3 = -1$ we have $L = -I = -R_{u_1,0}$.

Now suppose that A has one real eigenvalue λ and a pair of conjugate eigenvalues $\mu = a + ib$ and $\overline{\mu} = a - ib$. Since $A^*A = A^tA = I$ we know that A is unitarily diagonalizable over $\mathbf{C}$ and its eigenvalues all have length 1. Since $|\lambda| = 1$ we have $\lambda = \pm 1$ and since $|\mu| = |a + ib| = 1$ we have $\mu = e^{i\,\theta}$, that is $a = \cos\theta$ and $b = \sin\theta$, for some angle θ. By following our usual procedure for finding an orthonormal basis of eigenvectors, we can find an orthonormal basis

$$\mathcal{V} = \{u, w, \overline{w}\}$$

for $\mathbf{C}^3$ where $u \in \mathbf{R}^3$ is a real unit eigenvector for the real eigenvalue λ and $w = x + i\,y$, with $x, y \in \mathbf{R}^3$, is a complex unit eigenvector for the complex eigenvalue μ. In this basis we have $\big[L\big]_{\mathcal{V}} = \text{diag}\big(\lambda, w, \overline{w}\big)$. As in question 3 of assignment 4, but with an additional scaling factor, we let

$$\mathcal{U} = \big\{u, \sqrt{2}\,x, \sqrt{2}\,y\big\}\,.$$

We claim that $\mathcal{U}$ is an orthonormal basis for $\mathbf{R}^3$. We use the fact that $\mathcal{V}$ is an orthonormal basis for $\mathbf{C}^3$. We have $|u| = 1$ and we have $|w| = 1$ so that $|x|^2 + |y|^2 = |w|^2 = 1$. Since $\langle u, w\rangle = 0$ we have

$$0 = \langle u, w\rangle = \langle u, x + i\,y\rangle = \langle u, x\rangle - i\,\langle u, y\rangle = (u \cdot x) - i\,(u \cdot y)$$

and so $u \cdot x = u \cdot y = 0$. Since $\langle w, \overline{w}\rangle = 0$ we have

$$0 = \langle w, \overline{w}\rangle = \langle x + i\,y, x - i\,y\rangle = \langle x, x\rangle + i\,\langle x, y\rangle + i\,\langle y, x\rangle - \langle y, y\rangle = \big(|x|^2 - |y|^2\big) + i\big(2(x \cdot y)\big)$$

and so $|x|^2 = |y|^2$ and $x \cdot y = 0$. Finally, since $|x|^2 = |y|^2$ and $|x|^2 = |y|^2$ we have $|x| = |y| = \frac{1}{2}$ so that $|\sqrt{2}\,x| = |\sqrt{2}\,y| = 1$. Thus $\mathcal{U}$ is an orthonormal basis for $\mathbf{R}^3$, as claimed.

Note that (as we saw in question 3 of assignment 4), since $Au = \lambda u$ and $Aw = \mu w$ so that

$$\begin{aligned} A(x + i\,y) &= (a + i\,b)(x + i\,y) \\ Ax + i\,Ay &= (ax - by) + i\,(bx + ay) \\ Ax &= ax - by\;,\;\; Ay = bx + ay \end{aligned}$$

we have

$$\big[L\big]_{\mathcal{U}} = \begin{pmatrix} \lambda & & \\ & a & b \\ & -b & a \end{pmatrix} = \begin{pmatrix} \pm 1 & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \pm 1 & & \\ & \cos(-\theta) & -\sin(-\theta) \\ & \sin(-\theta) & \cos(-\theta) \end{pmatrix}.$$

When $\lambda = 1$ this is a rotation and when $\lambda = -1$ it is a rotary inversion.

We remark that, as in question 1(b) of Assignment 4, some care is required in determining u and ϕ such that $L = \pm R_{u,\phi}$. These depend in part on whether the basis $\mathcal{U}$ is positively or negatively oriented. When $\mathcal{U}$ is positively oriented, if $\lambda = 1$ then we can take $u = u_1$ and $\phi = -\theta$ to get $L = R_{u,\phi}$, and if $\lambda = -1$ then we can take $u = u_1$ and $\phi = \pi - \theta$ to get $L = -R_{u,\phi}$.