MATH 245 Linear Algebra 2, Solutions to Assignment 6

1: (a) For the quadratic curve $7x^2+8xy+y^2+5=0$, find the coordinates of each vertex, find the equation of each asymptote, and sketch the curve.

Solution: Let $K(x,y)=7x^2+8xy+y^2$. Note that $K(x,y)=\begin{pmatrix}x & y\end{pmatrix}A\begin{pmatrix}x\\y\end{pmatrix}$ where $A=\begin{pmatrix}7&4\\4&1\end{pmatrix}$. The characteristic polynomial of A is

$$\left|A-tI\right|=\begin{vmatrix}7-t&4\\4&1-t\end{vmatrix}=t^2-8t-9=(t-9)(t+1)$$

so the eigenvalues are $\lambda_1=9$ and $\lambda_2=-1$. For $\lambda=9$ we have

$$A-\lambda I=\begin{pmatrix}-2&4\\4&-8\end{pmatrix}\sim\begin{pmatrix}1&-2\\0&0\end{pmatrix}$$

so we can choose the unit eigenvector $u_1=\frac{1}{\sqrt5}\begin{pmatrix}2\\1\end{pmatrix}$ for λ_1. The other eigenspace will be orthogonal since A is symmetric, so we can choose the unit eigenvector $u_2=\frac{1}{\sqrt5}\begin{pmatrix}-1\\2\end{pmatrix}$ for λ_2. Let $P=(u_1,u_2)=\frac{1}{\sqrt5}\begin{pmatrix}2&-1\\1&2\end{pmatrix}$ and let $D=\begin{pmatrix}9&0\\0&-1\end{pmatrix}$. Then we have $P^*AP=D$. Write $\begin{pmatrix}x\\y\end{pmatrix}=P\begin{pmatrix}s\\t\end{pmatrix}$ or equivalently $\begin{pmatrix}s\\t\end{pmatrix}=P^*\begin{pmatrix}x\\y\end{pmatrix}$. Then

$$K(x,y)=\begin{pmatrix}x & y\end{pmatrix}A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}s & t\end{pmatrix}P^*AP\begin{pmatrix}s\\t\end{pmatrix}=\begin{pmatrix}s & t\end{pmatrix}D\begin{pmatrix}s\\t\end{pmatrix}=9s^2-t^2$$

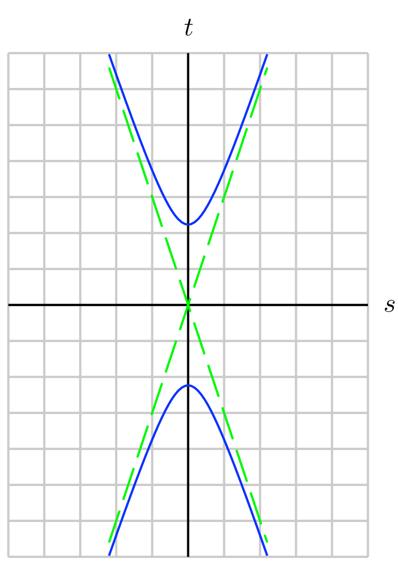
and so

$$K(u,v)=-5\iff 9s^2-t^2=-5\iff\frac{t^2}{5}-\frac{s^2}{5/9}=1\,.$$

This is the hyperbola in the st-plane with vertices at $(0,\pm\sqrt5)$ and asymptotes $t=\pm3s$. We calculate the points (x,y) corresponding to $(s,t)=(0,\pm\sqrt5)$ (the vertices) and $(s,t)=(\sqrt5,\pm3\sqrt5)$ (points on the asymptotes):

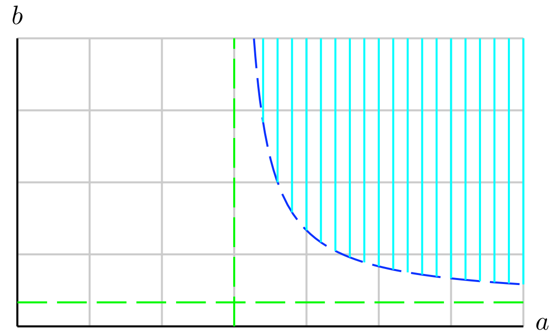
$$P\begin{pmatrix}0\\\pm\sqrt5\end{pmatrix}=\pm\begin{pmatrix}2&-1\\1&2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\pm\begin{pmatrix}-1\\2\end{pmatrix}$$
$$P\begin{pmatrix}\sqrt5\\\pm3\sqrt5\end{pmatrix}=\begin{pmatrix}2&-1\\1&2\end{pmatrix}\begin{pmatrix}1\\\pm3\end{pmatrix}=\begin{pmatrix}-1\\7\end{pmatrix},\begin{pmatrix}5\\-5\end{pmatrix}\,.$$

Thus the curve $K(x,y)=-5$ is a hyperbola with vertices at $(x,y)=(-1,2)$ and $(1,-2)$ and asymptotes $y=-x$ and $y=-7x$.



(b) For the real quadratic form $K(x,y,z) = 3x^2 + ay^2 + bz^2 - 6xy + 2xz - 4yz$, sketch the set of points (a,b) for which K is positive-definite.

Solution: We have $K(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where $A = \begin{pmatrix} 3 & -3 & 1 \\ -3 & a & -2 \\ 1 & -2 & b \end{pmatrix}$. Note that $\det A^{1\times 1} = \det(3) = 3$, $\det A^{2\times 2} = \det \begin{pmatrix} 3 & -3 \\ -3 & a \end{pmatrix} = 3a - 9$ and $\det A^{3\times 3} = \det A = 3ab + 6 + 6 - 12 - 9b - a = 3ab - 9b - a$. For K to be positive-definite we need $\det A^{2\times 2} > 0$, that is $3a - 9 > 0$ so $a > 3$, and we need $\det A^3 > 0$, that is $3ab - 9b - a > 0$ or equivalently $3b(a-3) > a$. For $a > 3$ this gives $b > \frac{a}{3(a-3)} = \frac{1}{3} + \frac{1}{a-3}$. Thus the required set of points (a,b) lies above and to the right of the hyperbola $y = \frac{1}{3} + \frac{1}{a-3}$, $a > 3$.



2: Let U and V be non-trivial subspaces of $\mathbf{R}^n$ with $U \cap V = \{0\}$. Recall that

$$\text{angle}\,(U,V) = \min\left\{\text{angle}\,(u,v) \middle| 0 \neq u \in U, 0 \neq v \in V\right\}.$$

(a) Show that $\text{angle}\,(U,V) = \cos^{-1}(\sigma)$ where σ is the largest singular value of the linear map $P : U \to V$ given by $P(x) = \text{Proj}_V(x)$.

Solution: Recall from Assignment 1 that for fixed $0 \neq u \in \mathbf{R}^n$ we have $\min\limits_{0\neq v\in V} \text{angle}\,(u,v) = \cos^{-1}\left|\text{Proj}_V \frac{u}{|u|}\right|$. Thus we have

$$\begin{aligned}
\text{angle}\,(U,V) &= \min_{0\neq u\in U} \min_{0\neq v\in V} \text{angle}\,(u,v) \\
&= \min_{0\neq u\in U} \cos^{-1}\left(\left|\text{Proj}_V \tfrac{u}{|u|}\right|\right) \\
&= \cos^{-1}\left(\max_{0\neq u\in U}\left|\text{Proj}_V \tfrac{u}{|u|}\right|\right) \\
&= \cos^{-1}\left(\max_{u\in U, |u|=1}\left|\text{Proj}_V(u)\right|\right) \\
&= \cos^{-1}\left(\max_{u\in U, |u|=1}\left|P(u)\right|\right) \\
&= \cos^{-1}\sigma
\end{aligned}$$

where σ is the largest singular value of P.

(b) Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Let $U = \operatorname{Span}\{u_1, u_2\}$ and $V = \operatorname{Span}\{v_1, v_2\}$. Find $\operatorname{angle}(U, V)$.

Solution: To find the singular values of P we find the matrix of P^*P with respect to orthonormal bases. We apply the Gram-Schmidt Procedure to the basis $\{u_1, u-2\}$ to get

$$w_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \;,\; w_2 = u_2 - \frac{u_2 \bullet w_1}{|w_1|^2}\, w_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 2 \end{pmatrix} - \frac{6}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 4 \\ 2 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -3 \\ 3 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

then we normalize to get the orthonormal basis $\mathcal{U} = \{x_1, x_2\}$ with

$$x_1 = \frac{w_1}{|w_1|} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \;,\; x_2 = \frac{w_2}{|w_2|} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

Next we apply the Gram-Schmidt Procedure to $\{v_1, v_2\}$ to get

$$z_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \;,\; z_2 = v_2 - \frac{v_2 \bullet z_1}{|z_1|^2}\, z_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{3}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

then we normalize to get the orthonormal basis $\mathcal{V} = \{y_1, y_2\}$ for V where

$$y_1 = \frac{z_1}{|z_1|^2} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \;,\; y_2 = \frac{z_2}{|z_2|^2} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Now let

$$A = [P]_{\mathcal{V}}^{\mathcal{U}} = \begin{pmatrix} x_1 \bullet y_1 & x_2 \bullet y_1 \\ x_1 \bullet y_2 & x_2 \bullet y_2 \end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

Then

$$[P^*P]_{\mathcal{V}}^{\mathcal{U}} = A^*A = \frac{1}{12}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 2 & 2 \\ 2 & 10 \end{pmatrix} = \tfrac{1}{6}\, B \;,\; \text{where } B = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}.$$

The characteristic polynomial of B is

$$\left|B - tI\right| = \begin{pmatrix} 1-t & 1 \\ 1 & 5-t \end{pmatrix} = t^2 - 6t + 4$$

so B has eigenvalues $\lambda = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$. Since $A^*A = \frac{1}{6}\, B$, the eigenvalues of A^*A, or equivalently the eigenvalues of P^*P, are $\frac{3 \pm \sqrt{5}}{6}$. Thus the largest singular value of P is $\sigma = \sqrt{\frac{3+\sqrt{5}}{6}}$ and we obtain

$$\operatorname{angle}(U, V) = \cos^{-1}\sqrt{\tfrac{3+\sqrt{5}}{6}}\,.$$

3: Let $\mathbf{F} = \mathbf{Z}_7$, the field of integers modulo 7.

(a) Let $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 4 & 3 \\ 5 & 3 & 0 \end{pmatrix} \in M_{3\times 3}(\mathbf{F})$. Find $Q \in GL(3, \mathbf{F})$ such that $Q^t A Q$ is diagonal.

Solution: We use column and row operations to put A into diagonal form. At each stage we indicate the operations used and give the elementary matrix for the column operations.

$$C_2 \mapsto C_2 + 3C_1 \quad \begin{pmatrix} 2 & 0 & 5 \\ 1 & 0 & 3 \\ 5 & 4 & 0 \end{pmatrix} \quad R_2 \mapsto R_2 + 3R_1 \quad \begin{pmatrix} 2 & 0 & 5 \\ 0 & 0 & 4 \\ 5 & 4 & 0 \end{pmatrix} \quad E_1 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_3 \mapsto C_3 + C_1 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 5 & 4 & 5 \end{pmatrix} \quad R_3 \mapsto R_3 + R_1 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 5 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_2 \mapsto C_2 + C_3 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 2 & 5 \end{pmatrix} \quad R_2 \mapsto R_2 + R_3 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C_3 \mapsto C_3 + 2C_2 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 2 & 2 \end{pmatrix} \quad R_3 \mapsto R_3 + 2R_2 \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus we can take

$$\begin{aligned} Q = E_1E_2E_3E_4 &= \left(\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}. \end{aligned}$$

(b) Find the number of distinct congruence classes of 3×3 symmetric matrices over $\mathbf{F}$.

Solution: We claim that there are 7 congruence classes. Indeed we claim more generally that for each positive integer n there are $2n+1$ congruence classes of symmetric $n \times n$ matrices over $\mathbf{F}$. There is only one $n \times n$ matrix with rank zero, namely the zero matrix. We shall show that for $1 \le r \le n$, every symmetric $n \times n$ matrix of rank r over $\mathbf{F}$ is congruent to exactly one of the two matrices

$$\begin{pmatrix} I_r & \\ & 0_{n-r} \end{pmatrix}, \quad \begin{pmatrix} 3 & & \\ & I_{r-1} & \\ & & 0_{n-r} \end{pmatrix}.$$

Let $1 \le r \le n$ and let $A \in M_{n\times n}(\mathbf{F})$ with $A^t = A$ and $\text{rank}(A) = r$. We know that A is congruent to a diagonal matrix $D = \text{diag}(d_1, \cdots, d_n)$. Note that exactly r of the entries d_i will be non-zero since $\text{rank}(D) = \text{rank}(A) = r$. In $\mathbf{F} = \mathbf{Z}_7$ we have the following table of squares.

x	0	1	2	3	4	5	6
x^2	0	1	4	2	2	4	1

We group the non-zero elements into two types, the squares $\{1,2,4\}$ and the non-squares $\{3,5,6\}$. Using the column and row operations $C_i \leftrightarrow C_j$, $R_i \leftrightarrow R_j$ we can rearrange the entries d_i of D. We order them so that $d_1, \cdots, d_k \in \{3,5,6\}$, $d_{k+1}, \cdots, d_r \in \{1,2,4\}$ and $d_{r+1}, \cdots, d_n = 0$. Define $f : \mathbf{F} \to \mathbf{F}$ by

x	0	1	2	3	4	5	6
$f(x)$	1	1	2	1	3	3	2

so that we have

$$x\,f(x)^2 = \begin{cases} 3 \text{ for } x \in \{3,5,6\} \\ 1 \text{ for } x \in \{1,2,4\} \\ 0 \text{ for } x = 0 \end{cases}$$

and let Q be the diagonal matrix $Q = \text{diag}\big(f(d_1), \cdots, f(d_n)\big)$. Note that Q is invertible since each $f(d_i) \ne 0$, and A is congruent to the matrix

$$\begin{aligned} E = Q^t D Q &= \text{diag}\big(d_1 f(d_1)^2, \cdots, d_n f(d_n)^2\big) \\ &= \text{diag}\big(3, \cdots, 3, 1, \cdots, 1, 0, \cdots, 0\big) \\ &= \begin{pmatrix} 3I_k & & \\ & I_{r-k} & \\ & & 0_{n-r} \end{pmatrix}. \end{aligned}$$

Next we note that

$$\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}^t \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so we have $3I_2$ congruent to I_2. It follows that, up to congruence, we can replace copies of the 2×2 block $3I_2$ in the above matrix E by copies of I_2, and hence when k is even A is congruent to $\begin{pmatrix} I_r & \\ & 0_{n-r} \end{pmatrix}$ and when k is odd A is congruent to $\begin{pmatrix} 3 & & \\ & I_{r-1} & \\ & & 0_{n-r} \end{pmatrix}$. Finally, we must show that these two matrices are not congruent. Suppose, for a contradiction, that they are. Say

$$\begin{pmatrix} S & T \\ U & V \end{pmatrix}^t \begin{pmatrix} I_r & \\ & 0 \end{pmatrix} \begin{pmatrix} S & T \\ U & V \end{pmatrix} = \begin{pmatrix} 3 & & \\ & I_{r-1} & \\ & & 0_{n-r} \end{pmatrix}$$

where S is of size $r \times r$. Then we have

$$\begin{pmatrix} S^t & U^t \\ T^t & R^t \end{pmatrix} \begin{pmatrix} S & T \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & & \\ & I_{r-1} & \\ & & 0_{n-r} \end{pmatrix}$$

$$\begin{pmatrix} S^t S & S^t T \\ T^t S & T^t T \end{pmatrix} = \begin{pmatrix} 3 & & \\ & I_{r-1} & \\ & & 0_{n-r} \end{pmatrix}$$

and so $S^t S = \begin{pmatrix} 3 & \\ & I_{r-1} \end{pmatrix}$. This is impossible since $\det(S^t S) = (\det S)^2 \in \{1,2,4\}$, but $\det \begin{pmatrix} 3 & \\ & I_{r-1} \end{pmatrix} = 3$.

4: (a) Let $A = \begin{pmatrix} 1-i & i \\ 2i & -1+i \end{pmatrix} \in M_{2\times 2}(\mathbf{C})$. Find $\max_{|x|=1} \big|Ax\big|$ and $\min_{|y|=1} \big|Ay\big|$, and find unit vectors x and y for which these maximum and minimum values are attained.

Solution: In class, we showed how to find $\max_{|u|=1} |L(u)|$ and $\min_{|u|=1} |L(u)|$ when L is a linear map of finite-dimensional vector spaces over $\mathbf{R}$. The same argument applies when U and V are finite-dimensional vector spaces over $\mathbf{C}$. We find that $\max_{|u|=1} |L(u)| = \sigma_1$ with $|L(u_1)| = \sigma_1$ and $\min_{|u|=1} |L(u)| = \sigma_n$ with $L(u_n) = \sigma_n$ where $\sigma_1 \leq \cdots \leq \sigma_n$ are the singular values of L, that is the square roots of the eigenvalues of L^*L) and $u_1, \cdots, u_n$ are corresponding orthonormal eigenvectors of L^*L. We have

$$A^*A = \begin{pmatrix} 1+i & -2i \\ -i & -1-i \end{pmatrix} \begin{pmatrix} 1-i & i \\ 2i & -1+i \end{pmatrix} = \begin{pmatrix} 6 & 1+3i \\ 1-3i & 3 \end{pmatrix} .$$

The characteristic polynomial of A^*A is

$$\big|A^*A - tI\big| = \begin{vmatrix} 6-t & 1+3i \\ 1-3i & 3-t \end{vmatrix} = t^2 - 9t + 8 = (t-8)(t-1)$$

so the eigenvalues of A^*A are $\lambda_1 = 8$ and $\lambda_2 = 1$, hence the singular values of A are $\sigma_1 = 2\sqrt{2}$ and $\sigma_2 = 1$. For $\lambda = 8$ we have

$$A^*A - \lambda I = \begin{pmatrix} -2 & 1+3i \\ 1-3i & -5 \end{pmatrix} \sim \begin{pmatrix} -2 & 1+3i \\ 0 & 0 \end{pmatrix}$$

so we can choose $u_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 1+3i \\ 2 \end{pmatrix}$ as a unit eigenvector for λ_1. Since the other eigenspace is orthogonal we can, by inspection, choose $u_2 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}$ as a unit eigenvector for λ_2. Thus $\max_{|x|=1} \big|Ax\big| = \sigma_1 = 2\sqrt{2}$ with this maximum attained when $x = u_1$, and $\min_{|y|=1} \big|Ay\big| = 1$ with this minimum attained when $y = u_2$.

(b) Let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Let $A \in M_{n\times n}(\mathbf{F})$ with $A^* = A$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A, listed in increasing order, with repetition according to multiplicity. Show that for each $k = 1, 2, \cdots, n$ we have

$$\lambda_k = \min_{U \subset \mathbf{F}^n, \dim U = k} \left(\max_{x \in U, |x|=1} x^* A\, x \right)$$

Solution: Since $A^* = A$, we know that A is unitarily diagonalizable. Choose a basis $\mathcal{U} = \{u_1, u_2, \cdots, u_n\}$ for $\mathbf{F}^n$ so that $Au_i = \lambda_i u_i$ for each i. Let U be any subspace of $\mathbf{F}^n$ with $\dim U = k$. Since the dimension of $\mathrm{Span}\{u_k, \cdots, u_n\}$ is equal to $n-k+1$, the intersection $U \cap \mathrm{Span}\{u_k, \cdots, u_n\}$ is non-trivial. Choose a unit vector $x \in U \cap \mathrm{Span}\{u_k, \cdots, u_n\}$, say $x = t_k u_k + \cdots + t_n u_n$. Then

$$\begin{aligned} x^*Ax &= (t_k u_k + \cdots + t_n u_n)^* A\, (t_k u_k + \cdots + t_n u_n) \\ &= (\overline{t_k}\, {u_k}^* + \cdots + \overline{t_n}\, {u_n}^*)(t_k \lambda_k u_k + \cdots + t_n \lambda_n u_n) \\ &= \lambda_k |t_k|^2 + \lambda_{k+1} |t_{k+1}|^2 + \cdots + \lambda_n |t_n|^2 \\ &\geq \lambda_k |t_k|^2 + \lambda_k |t_{k+1}^2| + \cdots + \lambda_k |t_n|^2 \\ &= \lambda_k \big(|t_k|^2 + \cdots + |t_n|^2\big) = \lambda_k |x|^2 = \lambda_k\, , \end{aligned}$$

so we have $\max_{x \in U, |x|=1} x^*Ax \geq \lambda_k$. Since this is true for every subspace $U \subset \mathbf{C}^n$ with $\dim U = k$, it follows that

$$\min_{U \subset \mathbf{C}^n, \dim U = k} \left(\max_{x \in U, |x|=1} x^*Ax \right) \geq \lambda_k\, .$$

Finally, note that for the k-dimensional subspace $U = \mathrm{Span}\{u_1, \cdots, u_k\}$ we know that $\max_{x \in U, |x|=1} x^*AX = \lambda_k$ since for $x = t_1 u_1 + \cdots t_k u_k$ with $|x| = 1$ we have $x^*Ax = \lambda_1 |t_1|^2 + \cdots + \lambda_n |t_n|^2 \leq \lambda_k \big(|t_1|^2 + \cdots + |t_n|^2\big) = \lambda_k$ with $x^*Ax = \lambda_k$ when $x = u_k$. Thus

$$\min_{U \subset \mathbf{C}^n, \dim U = k} \left(\max_{x \in U, |x|=1} x^*Ax \right) = \lambda_k\, .$$

5: Let U and V be vector spaces over a field $\mathbf{F}$ with $\mathrm{char}(\mathbf{F}) \neq 2$. For $u \in U$ and $v \in V$, let $u \otimes v$ denote the bilinear map from $U^* \times V^*$ to $\mathbf{F}$ given by

$$(u \otimes v)(f, g) = f(u)g(v)$$

for $f \in U^*$ and $g \in V^*$. For $u, v \in U$ let $u \odot v$ and $u \wedge v$ be the bilinear maps from $U^* \times U^* \to \mathbf{F}$ given by

$$u \odot v = \tfrac{1}{2}\big((u \otimes v) + (v \otimes u)\big) \,, \quad u \wedge v = \tfrac{1}{2}\big((u \otimes v) - (v \otimes u)\big) \,.$$

Note that $u \odot v$ is symmetric and $u \wedge v$ is alternating. The **tensor product** of U and V is defined to be

$$U \otimes V = \mathrm{Span}\left\{u \otimes v \middle| u \in U, v \in V\right\} \subset \mathrm{Bilin}(U^* \times V^*, \mathbf{F}) \,.$$

We define the spaces of 2-tensors, symmetric 2-tensors, and alternating 2-tensors on U to be

$$\begin{aligned} T^2U &= U \otimes U \\ S^2U &= \left\{S \in T^2U \,\middle|\, S \text{ is symmetric}\right\} \\ \Lambda^2U &= \left\{A \in T^2U \,\middle|\, A \text{ is alternating}\right\} \end{aligned}$$

Suppose that U and V are finite-dimensional, and let $\mathcal{U} = \{u_1, \cdots, u_n\}$ and $\mathcal{V} = \{v_1, \cdots, v_m\}$ be bases.
(a) Show that $\left\{u_i \otimes v_j \middle| 1 \leq i \leq n, 1 \leq j \leq m\right\}$ is a basis for $U \otimes V$ and that $U \otimes V = \mathrm{Bilin}(U^* \times V^*, \mathbf{F})$.

Solution: We begin by noting that for $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $c \in \mathbf{F}$ we have

$$(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v \,, \quad u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2 \,, \quad (cu) \otimes v = c(u \otimes v) = u \otimes (cv) \,.$$

To prove the first of the above three equalities, note that for all $f \in U^*$ and $g \in V^*$ we have

$$\begin{aligned} \big((u_1 + u_2) \otimes v\big)(f, g) &= f(u_1 + u_2)g(v) = \big(f(u_1) + f(u_2)\big)g(v) = f(u_1)g(v) + f(u_2)g(v) \\ &= (u_1 \otimes v)(f, g) + (u_2 \otimes v)(f, g) = \big((u_1 \otimes v) + (u_2 \otimes v)\big)(f, g) \,. \end{aligned}$$

The other two equalities are proven in the same way.

Let $\mathcal{W} = \left\{u_i \otimes v_j \middle| 1 \leq i \leq n, 1 \leq j \leq m\right\}$. Since each $u_i \otimes v_j \in U \otimes V$, we have $\mathrm{Span}\,\mathcal{W} \subset U \otimes V$. To show that $U \otimes V \subset \mathrm{Span}\,\mathcal{W}$ it suffices to show that for all $u \in U$ and $v \in V$ we have $u \otimes v \in \mathrm{Span}\,\mathcal{W}$, and indeed for $u = \sum_{i=1}^{n} s_i u_i \in U$ and $v = \sum_{j=1}^{m} t_j v_j \in V$ we have

$$u \otimes v = \Big(\sum_{i=1}^{n} s_i u_i\Big) \otimes \Big(\sum_{j=1}^{m} t_j v_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m} s_i t_j (u_i \otimes v_j) \in \mathrm{Span}\,\mathcal{W} \,.$$

Thus $\mathcal{W}$ spans $U \otimes V$. To show that $\mathcal{W}$ is linearly independent, suppose that $\sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j} u_i \otimes v_j = 0$. This means that $\Big(\sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j} u_i \otimes v_j\Big)(f, g) = 0$ for all $f \in U^*$ and $g \in V^*$. Let $\mathcal{F} = \{f_1, \cdots, f_n\}$ and $\mathcal{G} = \{g_1, \cdots, g_m\}$ be the bases for U^* and V^* which are dual to $\mathcal{U}$ and $\mathcal{V}$. Then for $1 \leq k \leq n$ and $1 \leq l \leq m$ we have

$$\begin{aligned} 0 &= \Big(\sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j} u_i \otimes v_j\Big)(f_k, g_l) = 0 = \sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j} (u_i \otimes v_j)(f_k, g_l) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j} f_k(u_i) g_l(u_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j} \delta_{k,i} \delta_{l,j} = s_{k,l} \,. \end{aligned}$$

Thus $\mathcal{W}$ is linearly independent, and hence $\mathcal{W}$ is a basis for $U \otimes V$.

We claim that $U \otimes V = \mathrm{Bilin}\big(U^* \times V^*, \mathbf{F}\big)$. It suffices to show that every bilinear map $S : U^* \times V^* \to \mathbf{F}$ lies in $U \otimes V$. Let $S : U^* \times V^* \to \mathbf{F}$ be bilinear. Recall that S is completely determined by the values $S(f_i, g_j)$ where $f_i \in \mathcal{F}$, $g_j \in \mathcal{G}$, indeed $S\Big(\sum_{i=1}^{n} s_i f_i \,,\, \sum_{j=1}^{m} t_j g_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m} s_i t_j \, S(f_i, g_j)$. Let $T = \sum_{k=1}^{n}\sum_{l=1}^{m} S(f_k, g_l)(u_k \otimes v_l)$. Note that $T \in U \otimes V$. Also, for all $f_i \in \mathcal{F}$ and $g_j \in \mathcal{G}$ we have

$$T(f_i, g_j) = \sum_{k=1}^{n}\sum_{l=1}^{m} S(f_k, g_l)(u_k \otimes v_l)(f_i, g_j) = \sum_{k=1}^{n}\sum_{l=1}^{m} S(f_k, g_l)\delta_{k,i}\delta_{l,j} = S(f_i, g_j) \,.$$

Since $S(f_i, g_j) = T(f_i, g_j)$ for all $f_i \in \mathcal{F}$, $g_j \in \mathcal{G}$, we have $S = T$, and hence $S \in U \otimes V$.

(b) Show that $\{u_i \odot u_j \big| 1 \le i \le j \le n\}$ is a basis for S^2U.

Solution: Let $\mathcal{W} = \{u_i \odot u_j \big| 1 \le i \le j \le n\}$. We claim that $\mathcal{W}$ spans S^2U. Let $S \in S^2U$, that is $S \in U \otimes U$ and S is symmetric. From the last paragraph in the solution to part (a) we have $S = \sum_{i=1}^{n}\sum_{j=1}^{n} S(f_i, f_j)(u_i \otimes u_j)$. where $\mathcal{F} = \{f_1, \cdots, f_n\}$ is the basis for U^* which is dual to $\mathcal{U}$. Also, we have $S(f_i, f_j) = S(f_j, f_i)$ since S is symmetric, and so

$$\begin{aligned}
S &= \sum_{1 \le i,j \le n} S(f_i, f_j)(u_i \otimes u_j) \\
&= \sum_{i<j} S(f_i, f_j)(u_i \otimes u_j) + \sum_{i=j} S(f_i, f_j)(u_i \otimes u_j) + \sum_{i>j} S(f_i, f_j)(u_i \otimes u_j) \\
&= \sum_{i<j} S(f_i, f_j)(u_i \otimes u_j) + \sum_{i} S(f_i, f_i)(u_i \otimes u_i) + \sum_{i<j} S(f_j, f_i)(u_j \otimes u_i) \\
&= \sum_{i<j} S(f_i, f_j)\big((u_i \otimes u_j) + (u_j \otimes u_i)\big) + \sum_{i} S(f_i, f_i)(u_i \otimes f_i) \\
&= \sum_{i<j} 2S(f_i, f_j)(u_i \odot u_j) + \sum_{i} S(f_i, f_i)(u_i \odot u_i) \\
&\in \operatorname{Span} \mathcal{W}.
\end{aligned}$$

Next we claim that $\mathcal{W}$ is linearly independent. Suppose that $\sum_{1 \le i \le j \le n} s_{i,j}(u_i \odot u_j) = 0$. Then

$$\begin{aligned}
0 &= \sum_{i \le j} s_{i,j}(u_i \odot u_j) = \sum_{i \le j} \frac{s_{i,j}}{2}\big((u_i \otimes u_j) + (u_j \otimes u_i)\big) \\
&= \sum_{i \le j} \frac{s_{i,j}}{2}(u_i \otimes u_j) + \sum_{i \le j} \frac{s_{i,j}}{2}(u_j \otimes u_i) \\
&= \sum_{i<j} \frac{s_{i,j}}{2}(u_i \otimes u_j) + \sum_{i=j} \frac{s_{i,j}}{2}(u_i \otimes u_j) + \sum_{i=j} \frac{s_{i,j}}{2}(u_j \otimes u_i) + \sum_{i<j} \frac{s_{i,j}}{2}(u_j \otimes u_i) \\
&= \sum_{i<j} \frac{s_{i,j}}{2}(u_i \otimes u_j) + \sum_{i} s_{i,i}(u_i \otimes u_i) + \sum_{i>j} \frac{s_{j,i}}{2}(u_i \otimes u_j) \\
&= \sum_{i,j} t_{i,j}(u_i \otimes u_j)
\end{aligned}$$

where $t_{i,j} = \frac{s_{i,j}}{2}$ for $i < j$, and $t_{i,j} = \frac{s_{j,i}}{2}$ for $i > j$, and $t_{i,i} = s_{i,i}$. Since $\{u_i \otimes u_j \big| 1 \le i, j \le n\}$ is linearly independent, we must have $t_{i,j} = 0$ for all $1 \le i, j \le n$, and so $s_{i,j} = 0$ for all $1 \le i \le j \le n$.

(c) Show that $\{u_i \wedge u_j \big| 1 \le i < j \le n\}$ is a basis for $\Lambda^2 U$.

Solution: We omit the the solution to part (c) which is very similar to the solution to part (b).