Math 247, Fall Term 2011

Homework Assignment 1

Handed out on Wednesday, September 14; due on Wednesday, September 21

In Problem 1, let $\vec{x}$ and $\vec{y}$ be two non-zero vectors in $\mathbb{R}^n$. We consider the Cauchy-Schwarz inequality that was proved in class (Proposition 1.3):

$$\text{(C-S)} \qquad |\,\langle \vec{x}\,,\,\vec{y}\,\rangle\,| \ \leq ||\,\vec{x}\,|| \cdot ||\,\vec{y}\,||.$$

Problem 1. In the conditions described above, prove the following equivalence:

$$\left(\begin{array}{c} \text{(C-S) holds with} \\ \text{equality} \end{array} \right) \Leftrightarrow \Big(\exists\, \alpha \in \mathbb{R} \setminus \{0\} \text{ such that } \vec{y} = \alpha\vec{x} \Big).$$

For Problem 2 recall that besides the "usual" norm $||\,\vec{x}\,||$ of a vector $\vec{x} \in \mathbb{R}^n$ we have also introduced the "1-norm" and the "∞-norm" of $\vec{x}$. They are defined by

$$||\,\vec{x}\,||_1 := \sum_{i=1}^{n} |\,x^{(i)}\,| \quad \text{and} \quad ||\,\vec{x}\,||_\infty := \max\{\,|\,x^{(1)}\,|,\ |\,x^{(2)}\,|, \ldots, |\,x^{(n)}\,|\,\},$$

where $x^{(1)}, \ldots, x^{(n)}$ are the components of $\vec{x}$.

Problem 2. Prove that for every $\vec{x} \in \mathbb{R}^n$ one has the following inequalities:

$$||\,\vec{x}\,||_\infty \leq ||\,\vec{x}\,|| \leq ||\,\vec{x}\,||_1 \leq n \cdot ||\,\vec{x}\,||_\infty.$$

Problem 3 asks you to prove an inequality satisfied by linear transformations. Let $M = [a_{i,j}]_{i,j}$ be an $m \times n$ matrix with real entries. Recall from Math 146 that one associates to M a linear transformation $T_M : \mathbb{R}^n \to \mathbb{R}^m$ defined as follows: if $\vec{x} = (\,x^{(1)}, \ldots, x^{(n)}\,) \in \mathbb{R}^n$, then $T_M(\,\vec{x}\,) := \vec{y} = (\,y^{(1)}, \ldots, y^{(m)}\,) \in \mathbb{R}^m$, where we put

$$y^{(i)} = \sum_{j=1}^{n} a_{i,j} x^{(j)}, \quad \forall\, 1 \leq i \leq m.$$

In other words: we have the formula "$T_M(\,\vec{x}\,) = M \cdot \vec{x}$", where we view $\vec{x}$ as an $n \times 1$ matrix and we use the rules for matrix multiplication.

Problem 3. Let M be as above, and consider the number $C := \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}^2}$. Prove that the linear transformation $T_M : \mathbb{R}^n \to \mathbb{R}^m$ satisfies the following inequality:

$$||T_M(\,\vec{x}\,)|| \leq C \cdot ||\,\vec{x}\,||, \quad \forall\, \vec{x} \in \mathbb{R}^n.$$

Problem 4. Write the proof for Proposition 2.5 in Lecture 2. The proposition stated that for a sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ we have the following equivalence:

$$\left(\begin{array}{c} (\vec{x}_k)_{k=1}^{\infty} \text{ is Cauchy} \\ \text{in } \mathbb{R}^n \end{array} \right) \Leftrightarrow \left(\begin{array}{c} \text{each of the component sequences} \\ (x_k^{(1)})_{k=1}^{\infty}, \ldots, (x_k^{(n)})_{k=1}^{\infty} \\ \text{is Cauchy in } \mathbb{R} \end{array} \right).$$

Problem 5. Let $(\vec{y}_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, and suppose that $\sum_{k=1}^{\infty} ||\vec{y}_k|| < \infty$ (where this infinite sum of non-negative real numbers is considered in the sense discussed in Math 148). For every $k \geq 1$ consider the vector $\vec{s}_k := \vec{y}_1 + \vec{y}_2 + \cdots + \vec{y}_k \in \mathbb{R}^n$. Prove that $(\vec{s}_k)_{k=1}^{\infty}$ is a convergent sequence in $\mathbb{R}^n$.

[Note: In the context of Problem 5, one says that the series $\sum_{k=1}^{\infty} \vec{y}_k$ is *absolutely convergent* in $\mathbb{R}^n$. The limit $\lim_{k\to\infty} \vec{s}_k$ is said to be *the sum* of this infinite series.]

Problem 6. (a) Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, and suppose there exist constants $c \in (0, \infty)$ and $\gamma \in (0, 1)$ such that

$$||\vec{x}_k - \vec{x}_{k+1}|| < c \cdot \gamma^k, \quad \forall\, k \geq 1. \tag{1}$$

Prove that the sequence $(\vec{x}_k)_{k=1}^{\infty}$ is convergent.

(b) Suppose the condition from Equation (1) of part (a) is replaced by

$$||\vec{x}_k - \vec{x}_{k+1}|| < 1/k^2, \quad \forall\, k \geq 1. \tag{2}$$

Can one still conclude that the sequence $(\vec{x}_k)_{k=1}^{\infty}$ is convergent? Justify your answer (proof or counterexample).

For Problem 7 recal that the *closed ball* centered at $\vec{a} \in \mathbb{R}^n$ and of radius $r > 0$ is denoted as $\overline{B}(\vec{a}; r)$ and is defined like this: $\overline{B}(\vec{a}; r) := \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x} - \vec{a}|| \leq r\}$.

Problem 7. In this problem $(\vec{a}_k)_{k=1}^{\infty}$ is a sequence of vectors in $\mathbb{R}^n$ and $r_1, r_2, \ldots, r_k, \ldots$ are strictly positive numbers with $\lim_{k\to\infty} r_k = 0$. Suppose that the closed balls $\overline{B}(\vec{a}_k; r_k)$ are *nested* inside each other, in the sense that we have the inclusions:

$$\overline{B}(\vec{a}_1; r_1) \supseteq \overline{B}(\vec{a}_2; r_2) \supseteq \cdots \supseteq \overline{B}(\vec{a}_k; r_k) \supseteq \cdots$$

(a) Prove that the sequence $(\vec{a}_k)_{k=1}^{\infty}$ is convergent.

(b) Let $\vec{a} := \lim_{k\to\infty} \vec{a}_k$. Prove that $\vec{a} \in \cap_{k=1}^{\infty} \overline{B}(\vec{a}_k; r_k)$.

(c) Could the intersection of closed balls $\cap_{k=1}^{\infty} \overline{B}(\vec{a}_k; r_k)$ contain some other vector, besides the vector $\vec{a}$ from part (b) of the problem? Justify your answer (proof or counterexample).