Math 247, Fall Term 2011

Homework assignment 2

Posted on Wednesday, September 21; due on Wednesday, September 28

Problem 1. Let α, β be two real numbers, and consider the sequences of real numbers $(s_k)_{k=1}^{\infty}$ and $(t_k)_{k=1}^{\infty}$ defined recursively as follows: $s_1 = 1$, $t_1 = 2$ and

$$\begin{cases} s_{k+1} & = & \alpha s_k + \beta t_k \\ t_{k+1} & = & \alpha t_k - \beta s_k \end{cases} , \quad \forall\, k \geq 1.$$

For every $k \geq 1$ we consider the vector $\vec{x}_k := (s_k, t_k) \in \mathbb{R}^2$.

(a) Prove that $||\, \vec{x}_{k+1} - \vec{x}_k \,|| = \sqrt{\alpha^2 + \beta^2} \cdot ||\, \vec{x}_k - \vec{x}_{k-1} \,||$, $\forall\, k \geq 2$.

(b) Suppose that $\alpha^2 + \beta^2 > 1$. Prove that the sequence $(\, \vec{x}_k \,)_{k=1}^{\infty}$ *is not* convergent.

(c) Suppose that $\alpha^2 + \beta^2 < 1$. Prove that the sequence $(\, \vec{x}_k \,)_{k=1}^{\infty}$ *is* convergent.

(d) Suppose that $\alpha^2 + \beta^2 = 1$. What can you say about the convergence of the sequence $(\, \vec{x}_k \,)_{k=1}^{\infty}$ in this case?

Let A be a subset of $\mathbb{R}^n$. Recall that the *interior* and the *closure* of a subset $A \subseteq \mathbb{R}^n$ were defined in class (Lecture 3, Definition 3.1) in the following way:

$$\text{int}(A) := \{\vec{a} \in A \mid \exists\, r > 0 \text{ such that } B(\vec{a}; r) \subseteq A\}, \quad \text{and}$$

$$\text{cl}(A) := \{\vec{x} \in \mathbb{R}^n \mid B(\vec{x}; r) \cap A \neq \emptyset, \quad \forall\, r > 0\}.$$

In your solution to Problems 2 and 3 please use the definitions stated above, and also the definition given in class for the concept of open set (a set $D \subseteq \mathbb{R}^n$ is defined to be *open* when it has the property that $\text{int}(D) = D$).

Problem 2. Let A be a subset of $\mathbb{R}^n$. By using the definitions of interior and closure that were stated above, prove the following equalities of sets.

(a) $\text{int}\big(\text{int}(A)\big) = \text{int}(A)$. (b) $\text{cl}\big(\text{cl}(A)\big) = \text{cl}(A)$.

Problem 3. Let A be a subset of $\mathbb{R}^n$. Your goal in this problem is to prove that "$\text{int}(A)$ is the *largest open set* sitting inside A". More precisely, you are required to prove the following two facts.

(a) Prove that $\text{int}(A)$ is an open set.

(b) Let D be an open subset of $\mathbb{R}^n$, such that $D \subseteq A$. Prove that $D \subseteq \text{int}(A)$.

Problem 4. Let A be a subset of $\mathbb{R}^n$. Your goal in this problem is to prove that "$\text{cl}(A)$ is the *smallest closed set* which contains A". More precisely, you are required to prove the following two facts.

(a) Prove that $\text{cl}(A)$ is a closed set.

(b) Let F be a closed subset of $\mathbb{R}^n$, such that $F \supseteq A$. Prove that $F \supseteq \text{cl}(A)$.

Problem 5. Let W be a linear subspace of $\mathbb{R}^n$, and suppose that $W \neq \mathbb{R}^n$. Prove that $\text{int}(W) = \emptyset$.

For Problem 6 recall that $\mathbb{Q}$ is the standard notation for the set of rational numbers. In the solution to this problem you are allowed to use the following facts:

(i) For every real number t, there exists a sequence $(r_k)_{k=1}^{\infty}$ of numbers from $\mathbb{Q}$ such that $\lim_{k\to\infty} r_k = t$.

(ii) For every real number t, there exists a sequence $(s_k)_{k=1}^{\infty}$ of numbers from $\mathbb{R} \setminus \mathbb{Q}$ such that $\lim_{k\to\infty} s_k = t$.

Problem 6. Consider the set $A = \{(s,t) \in \mathbb{R}^2 \mid s, t \in \mathbb{Q}\}$. Determine, with proof, what is the boundary $\text{bd}(A)$.

Definition. A subset $A \subseteq \mathbb{R}^n$ is said to be *sequentially compact* when it has the following property: for every sequence $(\,\vec{x}_k\,)_{k=1}^{\infty}$ of vectors from A one can find a convergent subsequence $(\,\vec{x}_{k(p)}\,)_{p=1}^{\infty}$ such that the limit $\vec{a} = \lim_{p\to\infty} \vec{x}_{k(p)}$ still belongs to A.

Problem 7. Let $A \subseteq \mathbb{R}^n$ be a sequentially compact set.

(a) Prove that A is closed.

(b) Prove that A is bounded.

Problem 8 uses the concept of *infimum* for a set of real numbers which is bounded from below. Before working on this problem, please review the concept of infimum (and the related concept of *supremum*) from Math 147 and 148.

Definition. Let A, B be two non-empty subsets of $\mathbb{R}^n$. The number

$$\beta := \inf\{\, ||\,\vec{x} - \vec{y}\,|| \;\mid\; \vec{x} \in A \text{ and } \vec{y} \in B\}$$

is called the *distance between the sets A and B*.

Problem 8. In this problem A and B are two nonempty subsets of $\mathbb{R}^n$, such that $A \cap B = \emptyset$. Let β be the distance between A and B, as defined above.

(a) Suppose that A is closed and that B is compact. Prove that $\beta > 0$.

(b) Suppose that A is closed and that B is compact. Prove that there exist $\vec{x}_o \in A$ and $\vec{y}_o \in B$ such that $||\,\vec{x}_o - \vec{y}_o\,|| = \beta$.

(c) In this part of the problem we only make the weaker assumption that A and B are closed. Show by example that it may happen that $\beta = 0$.