Math 247, Fall Term 2011

Homework assignment 3

Posted on Wednesday, September 28; due on Wednesday, October 5

Problem 1. Consider the following equivalence which was stated in class (in Lecture 4): given $A \subseteq \mathbb{R}^n$, a function $f : A \to \mathbb{R}^m$ and a point $\vec{a} \in A$, one has that

$$\left(\begin{array}{c} f \text{ respects the sequences in } A \\ \text{which converge to } \vec{a} \end{array} \right) \Leftrightarrow \left(\begin{array}{c} f \text{ is continuous at } \vec{a} \\ \text{in the sense of the } \varepsilon - \delta \text{ condition} \end{array} \right).$$

Write down the proof of the implication "$\Leftarrow$" of this equivalence.

Problem 2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\,(s,t)\,) = \begin{cases} \frac{4st^3}{s^2+t^2}, & \text{if } (s,t) \neq (0,0) \\ 0, & \text{if } (s,t) = (0,0). \end{cases}$$

Prove that f is continuous at $(0,0)$.

[In the solution to Problem 2 it is allowed to use without the proof the inequality

$$2\,|s| \cdot |t| \leq s^2 + t^2, \quad \forall\, s,t \in \mathbb{R},$$

which follows from the binomial formula.]

In Problems 3 and 4 we will use the following definition.

Definition. Let A be a nonempty subset of $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a function.
1^o The set $\Gamma := \{(\,\vec{x}\,, t) \in \mathbb{R}^{n+1} \mid \vec{x} \in A,\ t = f(x) \in \mathbb{R}\}$ is called the *graph* of f.
2^o The function f is said to be *bounded* when there exists $r > 0$ such that $|f(\,\vec{x}\,)| \leq r$, $\forall\, \vec{x} \in A$.

Problem 3. Let A be a closed subset of $\mathbb{R}^n$, let $f : A \to \mathbb{R}$ be a bounded function, and let $\Gamma \subseteq \mathbb{R}^{n+1}$ be the graph of f. Your goal in this problem is to prove the equivalence

$$\Big(f \text{ is continuous on } A\Big) \Leftrightarrow \Big(\Gamma \text{ is a closed subset of } \mathbb{R}^{n+1}\Big).$$

(a) Prove the implication "$\Rightarrow$" of the above equivalence.
(b) Prove the implication "$\Leftarrow$" of the above equivalence.

Problem 4. Let $A = \{(s,t) \in \mathbb{R}^2 \mid 1 \leq s \leq 2 \text{ and } 0 \leq t \leq 1\}$, and let $f : A \to \mathbb{R}$ be defined by the following formula:

$$f(\,(s,t)\,) := \begin{cases} s/t & \text{if } t \neq 0 \\ 0, & \text{if } t = 0. \end{cases}$$

(a) Let Γ be the graph of f. Prove that Γ is a closed subset of $\mathbb{R}^3$.

(b) Prove that f is not continuous on A. Does this contradict the implication "$\Leftarrow$" of the equivalence proved in Problem 3?

Definition. Let A be a subset of $\mathbb{R}^n$, let $f : A \to \mathbb{R}^m$ be a function and let p be a positive exponent. If there exists $c \geq 0$ such that

$$||f(\vec{x}) - f(\vec{y})|| \leq c \cdot ||\vec{x} - \vec{y}||^p, \quad \forall \, \vec{x}, \vec{y} \in A,$$

then one says that f is a *p-Lipschitz function.* In the special case when $p = 1$, one simply says that f is a *Lipschitz function.*

Problem 5. Let A be a subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ be a function. It is given that there exists an exponent $p > 0$ such that f is a p-Lipschitz function. Prove that f is uniformly continuous on A.

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function.
(a) Define what it means for f to be uniformly continuous on a subset $M \subseteq \mathbb{R}^2$.
(b) Consider the subsets $A, B \subseteq \mathbb{R}^2$ defined as follows:

$$A = \{(s,t) \in \mathbb{R}^2 \mid t \geq 0\}, \quad B = \{(s,t) \in \mathbb{R}^2 \mid t \leq 0\}.$$

Suppose that f is uniformly continuous on each of A and B. Does it follow that f is uniformly continuous on $\mathbb{R}^2$? Justify your answer.

The statement of Problem 7(c) goes under the name of "Banach's contraction principle".

Problem 7. Let A be a closed subset of $\mathbb{R}^n$. Let $f : A \to A$ be a function, and suppose there exists a constant $\gamma \in (0,1)$ with the property that

$$||f(\vec{x}) - f(\vec{y})|| \leq \gamma \cdot ||\vec{x} - \vec{y}||, \quad \forall \, \vec{x}, \vec{y} \in A.$$

(a) Let $\vec{x}_1$ be an arbitrary element of A. We define recursively

$$\vec{x}_2 := f(\vec{x}_1), \; \vec{x}_3 := f(\vec{x}_2), \ldots, \vec{x}_{k+1} := f(\vec{x}_k), \ldots$$

Prove that the sequence $(\vec{x}_k)_{k=1}^{\infty}$ is convergent, that the limit $\vec{p} := \lim_{k\to\infty} \vec{x}_k$ still belongs to A, and that $\vec{p}$ is a fixed point for the function f (in the sense that $f(\vec{p}) = \vec{p}$).

(b) By using his own methods (which he does not want to disclose) a certain Professor Mathinstock has found a point $\vec{q} \in A$ which also is a fixed point for f. Prove that $\vec{q}$ must coincide with the fixed point $\vec{p}$ from part (a) of the problem.

(c) By using parts (a) and (b) of the problem, prove that the function f has a unique fixed point in A.