Math 247, Fall Term 2011

Homework assignment 4

Posted on Wednesday, October 5; due on Wednesday, October 12

Problem 1. Let A be a nonempty subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ be a function. Let B be a subset of $\mathbb{R}^m$ such that $f(A) \subseteq B$, and let $g : B \to \mathbb{R}^p$ be a function. Consider the composed function $h = g \circ f$; that is, $h : A \to \mathbb{R}^p$ is defined by putting $h(\vec{x}) = g(f(\vec{x}))$ for every $\vec{x} \in A$. Suppose that f is uniformly continuous on A and g is uniformly continuous on B. Does it follow that h is uniformly continuous on A? Justify your answer (proof or counterexample).

In Problems 2-4 we will use the following definition.

Definition. Let A be a nonempty subset of $\mathbb{R}^n$.

1^o Let $\vec{x}$ be a vector in $\mathbb{R}^n$. The *distance from $\vec{x}$ to A* is the real non-negative number $d_A(\vec{x})$ defined as follows:

$$d_A(\vec{x}) \ := \ \inf\{ \, ||\vec{x} - \vec{a}|| \ \mid \vec{a} \in A\}.$$

2^o When the vector $\vec{x}$ from part 1^o of the definition is allowed to run in $\mathbb{R}^n$, we obtain a function $d_A : \mathbb{R}^n \to \mathbb{R}$. We will refer to it by calling it the *distance-to-A* function.

Problem 2. Let A be a nonempty subset of $\mathbb{R}^n$ and let $\vec{x}$ be a vector in $\mathbb{R}^n$. Prove the following equivalence:

$$\Big(d_A(\vec{x}) = 0 \Big) \Leftrightarrow \Big(\vec{x} \in \mathrm{cl}(A) \Big).$$

Problem 3. Let A be a nonempty subset of $\mathbb{R}^n$.

(a) Prove that for every $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ one has the inequality

$$d_A(\vec{x}) \leq d_A(\vec{y}) + ||\vec{x} - \vec{y}||.$$

(b) Prove that the distance-to-A function $d_A : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function. (The concept of Lipschitz function was defined in homework assignment 3.)

The following problem is an application of distance functions (it is recommended that you find the function f required in the problem by using a formula based on d_A and d_B).

Problem 4. Let A and B be closed nonempty subsets of $\mathbb{R}^n$, such that $A \cap B = \emptyset$. Prove that there exists a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ which has the following properties:

(i) $0 \leq f(\vec{x}) \leq 1$, for every $\vec{x} \in \mathbb{R}^n$;

(ii) $f(\vec{x}) = 0$ for every $\vec{x} \in A$; and

(iii) $f(\vec{x}) = 1$ for every $\vec{x} \in B$.

For Problems 5-7, recall the following notations (that were introduced in class, in Lecture 6). Let A be a subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded function. For a nonempty subset B of A we denote

$$\sup_B (f) := \sup\{f(\vec{x}) \mid \vec{x} \in B\}, \qquad \inf_B (f) := \inf\{f(\vec{x}) \mid \vec{x} \in B\},$$

and we also use the notation

$$\operatorname{osc}_B (f) := \sup_B (f) - \inf_B (f).$$

Problem 5. Let A be a subset of $\mathbb{R}^n$, let $f : A \to \mathbb{R}$ be a bounded function, and let B be a nonempty subset A. Prove that

$$\operatorname{osc}_B (f) = \sup\{\, |f(\vec{x}) - f(\vec{y})| \mid \vec{x}, \vec{y} \in B\}.$$

Problem 6. Let A be a subset of $\mathbb{R}^n$, let $f : A \to \mathbb{R}$ be a bounded function, and let B, C be nonempty subsets A such that $C \subseteq B$. Prove the following inequalities:

(a) $\sup_C (f) \leq \sup_B (f)$. (b) $\inf_C (f) \geq \inf_B (f)$. (c) $\operatorname{osc}_C (f) \leq \operatorname{osc}_B (f)$.

Problem 7. Let A be a subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded function. We fix a point $\vec{a} \in A$, and for every positive integer k we denote $N_k = A \cap B(\vec{a}; 1/k)$. Prove the following equivalence:

$$\left(\begin{array}{c} f \text{ is continuous} \\ \text{at } \vec{a} \end{array} \right) \Leftrightarrow \Big(\lim_{k \to \infty} \big(\operatorname{osc}_{N_k} (f) \big) = 0 \Big).$$