Math 247, Fall Term 2010

Homework assignment 5'

Posted on Wednesday, October 12; due on Wednesday, October 19

Note. The homework assignment 5 of Math 247 is divided into two parts. This is the first of the two parts. The second part (homework assignment 5") will be posted on the web-site of the course on Wednesday, October 26, and will be due in class on Wednesday, November 2.

Problem 1. Let P be a half-open rectangle in $\mathbb{R}^n$.

(a) Let Q be a half-open rectangle in $\mathbb{R}^n$, such that $Q \subseteq P$ and $Q \neq P$. Prove that one can find a grid-division $\Gamma = \{P_1, \ldots, P_r\}$ of P such that $Q = P_i$ for some $1 \leq i \leq r$.

(b) Let $Q_1, \ldots, Q_s$ be half-open rectangles in $\mathbb{R}^n$ such that $Q_i \subseteq P$ for every $1 \leq i \leq s$ and such that $Q_i \cap Q_j = \emptyset$ for every $1 \leq i < j \leq s$. Let us assume that the union $Q_1 \cup \cdots \cup Q_s$ is not equal to P. Prove that one can find some additional half-open rectangles $R_1, \ldots, R_t$ (for some $t \geq 1$) such that $\Delta = \{Q_1, \ldots, Q_s, R_1, \ldots, R_t\}$ is a division of P.

Problem 2. Consider the half-open rectangle $P = (0,1] \times (0,1] \subseteq \mathbb{R}^2$, and let $f : P \to \mathbb{R}$ be defined by $f(\,(s,t)\,) = s + t$ for every $(s,t) \in P$.

(a) Let k be a positive integer, and let Δ_k be the division of P into a grid of k^2 squares of side $1/k$. That is, $\Delta_k = \{P_{i,j} \mid 1 \leq i, j \leq k\}$, where for every $1 \leq i, j \leq k$ we define $P_{i,j} := ((i-1)/k, i/k] \times ((j-1)/k, j/k]$. Calculate the Darboux sums $L(f, \Delta_k)$ and $U(f, \Delta_k)$.

(b) For the divisions Δ_k from part (a), prove that $\lim_{k \to \infty} U(f, \Delta_k) - L(f, \Delta_k) = 0$.

(c) For the divisions Δ_k from part (a), calculate the limit $\lim_{k \to \infty} U(f, \Delta_k)$.

(d) Based on parts (b) and (c) of the problem, explain why the function f is integrable, and give the value of its integral.

Problem 3. Consider the half-open rectangle $P = (0,1] \times (0,1] \subseteq \mathbb{R}^2$, and the function $f : P \to \mathbb{R}$ defined as follows:

$$f(\,(s,t)\,) = \begin{cases} 1, & \text{if } s, t \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the lower integral $\underline{\int}_P f$ and the upper integral $\overline{\int}_P f$, and conclude that f is not integrable.

Problem 4. Let P be a half-open rectangle in $\mathbb{R}^n$ and let $f : P \to \mathbb{R}$ be a bounded function. Consider the function $-f$ (which acts by the formula $(-f)(\vec{x}) = -f(\vec{x})$, for every $\vec{x} \in P$).

(a) Prove that for every division Δ of P we have that $L(-f, \Delta) = -U(f, \Delta)$ and that $U(-f, \Delta) = -L(f, \Delta)$.

(b) Based on part (a) of the problem, prove that

$$\underline{\int}_P (-f) = -\overline{\int}_P f, \text{ and } \overline{\int}_P (-f) = -\underline{\int}_P f.$$

(c) Suppose that f is integrable. Based on part (b) of the problem, prove that $-f$ is integrable as well, and that $\int_P (-f) = -\int_P f$.