Math 247, Fall Term 2011

Homework assignment 6

Posted on Wednesday, November 2; due on Wednesday, November 9

Problem 1. Let A be the bounded subset of $\mathbb{R}^2$ defined as follows:

$$A := \{ (s,t) \in (0,1] \times (0,1] \mid s \leq t \}.$$

Consider the function $f : A \to \mathbb{R}$ defined by $f(\,(s,t)\,) := \sin(t^2)$, for $(s,t) \in A$.

(a) Prove that $\mathrm{bd}(A)$ is a null set of $\mathbb{R}^2$.

(b) Prove that the function f is integrable on A.

(c) By appropriately using the theorem of Fubini, calculate the integral $\int_A f$.

Problem 2. Let A be the bounded subset of $\mathbb{R}^3$ defined as follows:

$$A = \Big\{ (x,y,z) \in \mathbb{R}^3 \mid 0 < x \leq 1,\ 0 < y \leq 1 - x^2,\ 0 < z \leq x^2 + y \Big\}.$$

Consider the function $f : A \to \mathbb{R}$ defined by $f(\,(x,y,z)\,) := y$, for $(x,y,z) \in A$. We accept the fact f is integrable on A, and that the hypotheses of the theorem of Fubini are satisfied. By using the theorem of Fubini, calculate $\int_A f$.

Problem 3 shows an example of a function where hypothesis (ii) in the theorem of Fubini is satisfied, even though the hypothesis (i) is not. (Moral of this example: just knowing that it makes sense to talk about the iterated integral $\int_0^1 \Big(\int_0^1 f(\,(s,t)\,)\ dt \Big)\, ds$ will not automatically imply that f is integrable on $(0,1] \times (0,1]$.)

Problem 3. In this problem we accept the fact that the rational numbers can be enumerated as a sequence. We fix one of the possible ways of doing such an enumeration for the set of rationals in $(0,1]$:

$$\mathbb{Q} \cap (0,1] = \{q_1, q_2, \ldots, q_n, \ldots\}, \text{ where } q_m \neq q_n \text{ for } m \neq n.$$

Define a function $f : (0,1] \times (0,1] \to \mathbb{R}$ by the following formula:

$$f(\,(s,t)\,) = \begin{cases} 1, & \text{if } s = q_n \text{ and } t = q_m \text{ with } n \geq m \\ 0, & \text{otherwise.} \end{cases}$$

(a) Prove that the function f is not integrable on $(0,1] \times (0,1]$.

(b) For every $s \in (0,1]$ let $f_s : (0,1] \to \mathbb{R}$ be the partial function defined by $f_s(t) = f(\,(s,t)\,)$, $0 < t \leq 1$. Prove that every f_s is integrable, and that it makes sense to talk about the integral $\int_0^1 \Big(\int_0^1 f_s(t)\, dt \Big)\, ds$.

Problems 4–6 are geared towards calculating the volume of the standard simplex in $\mathbb{R}^n$.

Problem 4. (a) Let $P = (a_1, b_1] \times \cdots \times (a_n, b_n]$ be a half-open rectangle in $\mathbb{R}^n$. Let $\alpha > 0$ be a real number, and consider the new rectangle

$$Q = (\alpha a_1, \alpha b_1] \times \cdots \times (\alpha a_n, \alpha b_n] \subseteq \mathbb{R}^n.$$

Let $f : P \to \mathbb{R}$ be a bounded function, and let us define $g : Q \to \mathbb{R}$ by putting

$$g(\vec{y}) := f(\frac{1}{\alpha}\vec{y}), \quad \forall \vec{y} \in Q.$$

Prove that

$$\overline{\int_Q} g = \alpha^n \overline{\int_P} f \text{ and that } \underline{\int_Q} g = \alpha^n \underline{\int_P} f.$$

(b) In the setting of part (a) of the problem, let us assume that the function f is integrable on P. Prove that g is integrable on Q, with $\int_Q g = \alpha^n \int_P f$.

Problem 5. Let A be a bounded non-empty subset of $\mathbb{R}^n$, let $\alpha > 0$ be a real number, and consider the set

$$B := \{\alpha\,\vec{a} \mid \vec{a} \in A\} \subseteq \mathbb{R}^n.$$

Suppose that the set A has volume (in the sense of the definition given in homework assignment 5"). Prove that the set B has volume, and that $\mathrm{vol}(B) = \alpha^n \cdot \mathrm{vol}(A)$.

Definition. Let n be a positive integer. The set

$$S_n := \left\{ \vec{x} = \left(x^{(1)}, \ldots, x^{(n)}\right) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x^{(1)}, \ldots, x^{(n)} \geq 0 \text{ and} \\ x^{(1)} + \cdots + x^{(n)} \leq 1 \end{array} \right\}$$

is called the *standard simplex* in $\mathbb{R}^n$.

We accept the fact that, for every $n \geq 1$, the standard simplex S_n is a subset of $\mathbb{R}^n$ which has volume. Let V_n denote the volume of S_n.

Problem 6. Let S_n and V_n be as above. We accept the fact that the hypotheses of Fubini's theorem are satisfied in connection to the calculation of the volume V_n.

(a) Determine the values of V_1 and of V_2 (it is accepted to do this by using basic geometry).

(c) By using the theorem of Fubini, prove that $V_{n+1} = V_n/(n+1)$, $\forall\, n \geq 2$.

(d) Give a general formula for V_n, $n \geq 1$.