## Math 247, Fall Term 2011

## Homework assignment 5”

*Posted on Wednesday, October 26; due on Wednesday, November 2*

Problem 1 fills in a detail that remained from Lecture 10, in connection to the operation of taking the absolute value of a function. In your solution to Problem 1 you are allowed to use (without proof) the following elementary inequality satisfied by the absolute value function: one has

$$(*) \qquad \big|\ |s| - |t|\ \big| \leq |s-t|, \quad \forall\, s,t \in \mathbb{R}.$$

**Problem 1.** Let $P$ be a half-open rectangle in $\mathbb{R}^n$, and let $f : P \to \mathbb{R}$ be a function. Consider the new function $|f| : P \to \mathbb{R}$ which is defined by the formula

$$|f|(\vec{x}) := |\, f(\,\vec{x}\,)\,|, \quad \vec{x} \in P.$$

(a) Suppose that $f$ is bounded. Prove that $|f|$ is bounded as well, and that

$$\operatorname*{osc}_A\,(\,|f|\,) \leq \operatorname*{osc}_A\,(f), \quad \text{for every nonempty subset } A \subseteq P.$$

(b) Suppose that $f \in \mathrm{Int}_b(P,\mathbb{R})$ (i.e, $f$ is bounded and integrable on $P$). Prove that $|f| \in \mathrm{Int}_b(P,\mathbb{R})$ as well.

Problems 2 and 3 are about integrals on bounded subsets of $\mathbb{R}^n$, as discussed in class in Lecture 11.

**Problem 2.** Let $A_1, \ldots, A_r$ be nonempty bounded subsets of $\mathbb{R}^n$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Suppose that for every $1 \leq i \leq r$ we are given a function $f_i \in \mathrm{Int}_b(A_i,\mathbb{R})$ (that is, $f_i : A_i \to \mathbb{R}$ is bounded and integrable). Let $A := A_1 \cup \cdots \cup A_r$ and let $f : A \to \mathbb{R}$ be the function which is obtained by patching together the functions $f_1, \ldots, f_r$ (that is, $f(\,\vec{x}\,) = f_i(\,\vec{x}\,)$ for every $\vec{x} \in A_i$, $1 \leq i \leq r$). Prove that $f \in \mathrm{Int}_b(A,\mathbb{R})$, and that

$$\int_A f = \int_{A_1} f_1 + \cdots + \int_{A_r} f_r.$$

**Definition.** Let $A$ be a bounded subset of $\mathbb{R}^n$. We say that $A$ *has volume* when the constant function $1 : A \to \mathbb{R}$ is integrable on $A$; if this is the case, then we define the volume of $A$ by the formula

$$\mathrm{vol}(A) := \int_A 1.$$

**Remark.** $1^o$ Let $P = (a_1, b_1] \times \cdots \times (a_n, b_n]$ be a half-open rectangle in $\mathbb{R}^n$. It was observed in class (Lecture 10) that constant functions are integrable on $P$, and that

$$\int_P 1 = \mathrm{vol}(P) = (b_1 - a_1) \cdots (b_n - a_n).$$

Hence $P$ has volume in the sense of the above definition. Moreover, the formula for volume given in the above definition agrees with the "usual" formula defining the volume of $P$.

$2^o$ Let $A$ be a bounded subset of $\mathbb{R}^n$, and let $P$ be a half-open rectangle such that $A \subseteq P$. Consider the indicator function of $A$ relative to $P$, that is, the function $I_A : P \to \mathbb{R}$ which is defined by

$$I_A(\vec{x}) := \begin{cases} 1, & \text{if } \vec{x} \in A \\ 0, & \text{if } \vec{x} \in P \setminus A. \end{cases}$$

In view of how integrals on $A$ were defined in Lecture 11, we have that

$$\Big( A \text{ has volume } \Big) \Leftrightarrow \Big( I_A \in \mathrm{Int}_b(P, \mathbb{R}) \Big).$$

Moreover, if this happens, then the formula defining $\mathrm{vol}(A)$ takes the form

$$\mathrm{vol}(A) = \int_P I_A.$$

$3^o$ If $A = \emptyset \subseteq \mathbb{R}^n$, then one cannot really consider "the constant function 1 on $A$". We will nevertheless make the convention that $\emptyset$ has volume, and that $\mathrm{vol}(\emptyset) = 0$. This agrees with the formula given in part $2^o$ of the remark – indeed if $P$ is any half-open rectangle in $\mathbb{R}^n$, then the indicator function $I_\emptyset : P \to \mathbb{R}$ is just the zero-function on $P$, so we do have

$$\mathrm{vol}(\emptyset) = \int_P I_\emptyset = 0.$$

**Problem 3.** Let $A, B$ be two subsets of $\mathbb{R}^n$, such that both $A$ and $B$ have volume.
(a) Prove that $A \cap B$ has volume.
(b) Prove that $A \cup B$ has volume.
(c) Prove that $\mathrm{vol}(A \cup B) = \mathrm{vol}(A) + \mathrm{vol}(B) - \mathrm{vol}(A \cap B)$.

**Problem 4.** (a) Consider the set

$$E := \{(s,t) \in \mathbb{R}^2 \mid \frac{s^2}{4} + \frac{t^2}{9} = 1\} \subseteq \mathbb{R}^2.$$

($E$ is what we call an *ellipse*.) Prove that $E$ is a null subset of $\mathbb{R}^2$.

(b) Consider the set

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x, y, z \geq 0, \ x + y + z = 1\} \subseteq \mathbb{R}^3.$$

Prove that $C$ is a null subset of $\mathbb{R}^3$.