Math 247, Fall Term 2011

Homework assignment 7

Posted on Wednesday, November 9; due on Wednesday, November 16

Problem 1 fills in the proof of the fact stated in class (Lecture 14, Remark 14.6), that "one can change the values of a function f on a null set without affecting the integrability properties of f".

Problem 1. (a) Let N be a null subset of $\mathbb{R}^n$, and let $h : N \to \mathbb{R}$ be a bounded function. Prove that h is integrable on N, and that $\int_N h = 0$.

(b) Let P be a half-open rectangle in $\mathbb{R}^n$, let $f, g : P \to \mathbb{R}$ be two bounded functions, and suppose that the following conditions (i) and (ii) are fulfilled.
(i) There exists a null set $N \subseteq P$ such that $f(\vec{x}) = g(\vec{x})$ for every $\vec{x} \in P \setminus N$.
(ii) The function f is integrable on P.
Prove that g is integrable on P, and that $\int_P g = \int_P f$.

Problem 2 gives a generalization to n dimensions for the calculation done in class (Lecture 14, Remark 14.5) concerning the volume of the closed unit ball in $\mathbb{R}^3$. For every $n \in \mathbb{N}$, we accept the fact that the closed unit ball of $\mathbb{R}^n$ has volume, and we denote this volume by Ω_n. (For example, for $n = 1$ we have $\Omega_1 = 2$, because the closed unit ball in $\mathbb{R}$ is just the interval $[-1, 1]$. For $n = 2$ we have $\Omega_2 = \pi$, the area of a disc of radius 1 in the plane. The calculation from Remark 14.5 was done for $n = 3$, and showed that $\Omega_3 = 4\pi/3$.)

Problem 2. (a) In the notations introduced above, prove that

$$\Omega_{n+1}/\Omega_n = \int_{-1}^{1} (1 - r^2)^{n/2} \, dr, \quad \forall \, n \geq 1.$$

(c) By using the part (a) of the problem, determine the value of Ω_4.

[Note: when evaluating an integral of the form $\int_{-1}^{1} (1 - r^2)^{n/2} \, dr$, it may be convenient to use the trigonometric substitution $r = \sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$.]

Problem 3. Let b be a positive real number, and let D be the punctured disk of radius b in $\mathbb{R}^2$:

$$D = \{(s,t) \in \mathbb{R}^2 \mid 0 < \sqrt{s^2 + t^2} \leq b\}.$$

By using polar coordinates, calculate the integral $\int_D e^{-(s^2+t^2)} \, d(s,t)$.

Problem 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(\,(x,y)\,) := \begin{cases} x^2 y/(x^2 + y^2) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

(a) Consider the point $\vec{0} = (0,0) \in \mathbb{R}^2$. Prove that f has partial derivatives at $\vec{0}$, and calculate the values of $\big(\partial_1 f\big)(\vec{0})$ and $\big(\partial_2 f\big)(\vec{0})$.

(b) Consider again the point $\vec{0} \in \mathbb{R}^2$, and consider a vector $\vec{v} = (\alpha, \beta) \in \mathbb{R}^2$ where $\alpha \neq 0 \neq \beta$. Prove that the directional derivative $\big(\partial_{\vec{v}} f\big)(\vec{0})$ exists, and calculate its value.

(c) By using parts (a) and (b) of the problem, prove that one can consider the function $L : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$L(\vec{v}) = \big(\partial_{\vec{v}} f\big)(\vec{0}), \quad \vec{v} \in \mathbb{R}^2.$$

Is L a linear function? Justify your answer.

Problem 5. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(\,(x,y)\,) := \begin{cases} 1, & \text{if } x \neq 0 \text{ and } 0 < y < x^2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) On a picture of $\mathbb{R}^2$ mark the subset of the plane where g takes the value 0, and mark the subset of the plane where it takes the value 1.

(b) Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^2$. By using the picture drawn in part (a), prove that the directional derivative $(\partial_{\vec{v}}\, g)(\vec{0})$ exists and compute its value.

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function, and suppose that $(\partial_{\vec{v}} f)(\vec{0})$ exists for every $\vec{v} \in \mathbb{R}^2$. Does it follow that f is continuous at $\vec{0}$? Justify your answer (proof or counterexample).