Math 247, Fall Term 2011

Homework assignment 8

Posted on Wednesday, November 16; due on Wednesday, November 23

Problem 1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\,(x,y)\,) := \begin{cases} x^2y/(x^2+y^2) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

This is the same function as in Problem 4 of assignment 7. We accept the following fact, proved in assignment 7: f has partial derivatives at $\vec{0}$, and $\partial_1 f(\,\vec{0}\,) = \partial_2 f(\,\vec{0}\,) = 0$.

(a) Let $\vec{a} = (p,q)$ be a vector in $\mathbb{R}^2$, where $\vec{a} \neq \vec{0}$. Prove that f has partial derivatives at $\vec{a}$, and find the formulas for $\partial_1 f(\,\vec{a}\,)$ and $\partial_2 f(\,\vec{a}\,)$ as functions of p and q.

(b) Consider the function $\partial_1 f : \mathbb{R}^2 \to \mathbb{R}$. Is this function continuous? Justify your answer.

(c) Same question as in (b), in connection to the function $\partial_2 f : \mathbb{R}^2 \to \mathbb{R}$.

Problem 2. Let A be a non-empty open subset of $\mathbb{R}^n$, and let f be a function in $C^1(A,\mathbb{R})$. Suppose that $f(\,\vec{x}\,) \neq 0$, $\forall\, \vec{x} \in A$, and define $g : A \to \mathbb{R}$ by putting $g(\,\vec{x}\,) := 1/f(\,\vec{x}\,)$, $\vec{x} \in A$. Prove that $g \in C^1(A,\mathbb{R})$, and give a formula expressing the partial derivative $\partial_i g$ $(1 \leq i \leq n)$ in terms of f and $\partial_i f$.

Problem 3 asks you to fill in the proof for the statement of "MVT in direction $\vec{v}$" that was made in class (Lecture 16, Proposition 16.7). The proof is similar to the one showed in class for "MVT in the i-th direction" (Lemma 16.5), where the function of 1 variable that appears in the argument could now be $\psi : [0,1] \to \mathbb{R}$ defined by $\psi(s) = f(\,(1-s)\vec{x} + s\vec{y}\,)$.

Problem 3. Let A be an open subset of $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a continuous function. Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^n$, and suppose that the directional derivative $\big(\partial_{\vec{v}} f\big)(\,\vec{a}\,)$ is defined at every $\vec{a} \in A$. Let $\vec{x}, \vec{y}$ be two points in A such that the line segment $\mathrm{Co}(\,\vec{x}, \vec{y})$ is contained in A and such that $\vec{y} - \vec{x} = \alpha\vec{v}$ (for some $\alpha \in \mathbb{R}$). Prove that there exists a point $\vec{b} \in \mathrm{Co}(\,\vec{x}, \vec{y})$ such that

$$f(\,\vec{y}\,) - f(\,\vec{x}\,) = \alpha \cdot \big(\partial_{\vec{v}} f\big)(\,\vec{b}\,).$$

Definition. A set $A \subseteq \mathbb{R}^n$ is said to be *convex* when it has the following property: for every two points $\vec{a}, \vec{b} \in A$, the line segment $\mathrm{Co}(\vec{a}, \vec{b})$ is contained in A.

Problem 4. Let A be a nonempty open subset of $\mathbb{R}^n$ and let f be a function in $C^1(A,\mathbb{R})$. Let K be a nonempty subset of A which is compact and convex. Prove that f is Lipschitz on K; that is, prove there exists a constant $c > 0$ such that

$$|f(\,\vec{x}\,) - f(\,\vec{y}\,)| \leq c\,||\,\vec{x} - \vec{y}\,||, \quad \forall\, \vec{x}, \vec{y} \in K.$$

Problem 5. Let $A = \{(s,t) \in \mathbb{R}^2 \mid \frac{s^2}{4} + \frac{t^2}{9} < 1\}$ and let $f : A \to \mathbb{R}$ be the function defined by

$$f(\,(s,t)\,) = \sqrt{1 - \frac{s^2}{4} - \frac{t^2}{9}}, \quad (s,t) \in A.$$

It is accepted that A is an open subset of $\mathbb{R}^2$ and that $f \in C^1(A, \mathbb{R})$. Let $\Gamma \subseteq \mathbb{R}^3$ be the graph of f. Consider the point $\vec{a} = (1,2) \in A$, with $f(\,\vec{a}\,) = \sqrt{11}/6$, and consider the corresponding point $\vec{p} = (1, 2, \sqrt{11}/6) \in \Gamma$. Find a normal vector for Γ, at the point $\vec{p}$.

Definition. Let A be an open subset of $\mathbb{R}^n$, let $f : A \to \mathbb{R}$ be a function and let $\vec{a}$ be a point of A.

(a) We say that $\vec{a}$ is a point of *local minimum* for f if there exists $r > 0$ such that $B(\,\vec{a}\,; r) \subseteq A$ and such that $f(\,\vec{x}\,) \geq f(\,\vec{a}\,)$ for every $\vec{x} \in B(\,\vec{a}\,; r)$.

(b) We say that $\vec{a} \in A$ is a point of *local maximum* for f if there exists $r > 0$ such that $B(\,\vec{a}\,; r) \subseteq A$ and such that $f(\,\vec{x}\,) \leq f(\,\vec{a}\,)$ for every $\vec{x} \in B(\,\vec{a}\,; r)$.

Problem 6. Let A be an open subset of $\mathbb{R}^n$ and let f be a function in $C^1(A, \mathbb{R})$. Let $\vec{a}$ be a point of A which is either a local minimum or a local maximum for f. Prove that $(\nabla f)(\,\vec{a}\,)$ is the zero-vector of $\mathbb{R}^n$. (In other words, prove that $(\partial_i f)(\,\vec{a}\,) = 0$ for every $1 \leq i \leq n$.)

Problem 7. (a) Let $A = (-1,1) \times (-1,1) \subseteq \mathbb{R}^2$, and let $f : A \to \mathbb{R}$ be defined by the formula

$$f(\,(s,t)\,) = s^2 + 2st - t^2, \quad \forall\,(s,t) \in A.$$

Prove that f has no points of local minimum or local maximum on A.

(b) Let $B = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$, and let $g : B \to \mathbb{R}$ be defined by the formula

$$g(\,(s,t)\,) = s^2 + 2st - t^2, \quad \forall\,(s,t) \in B.$$

Due to EVT, g is bounded and must attain its global minimum and maximum on B. Based on part (a) of the question, explain why the global minimum and global maximum cannot be attained on the subset $A = (-1,1) \times (-1,1)$ of B.

(c) Determine the minimal and maximal value taken by the function g in part (b) of the question.