Math 247, Fall Term 2011

Homework assignment 9

Posted on Wednesday, November 23; due on Wednesday, November 30

The transformation $T : P \to \mathbb{R}^3$ discussed in Problem 1 is called "transformation to spherical coordinates in $\mathbb{R}^3$". The points in the domain P of this transformation are written in the form (r, ξ, θ), where ξ is called "angle of longitude" and θ is called "angle of latitude".

Problem 1. Let $P = (0,1) \times (0, 2\pi) \times (-\pi/2, \pi/2) \subseteq \mathbb{R}^3$ and consider the function $T : P \to \mathbb{R}^3$ defined by

$$T(\vec{x}) := (r \cos \xi \, \cos \theta, r \sin \xi \, \cos \theta, r \sin \theta), \quad \text{for } \vec{x} = (r, \xi, \theta) \in P.$$

(a) Prove that for a point $\vec{x} = (r, \xi, \theta) \in P$, one has $|| \, T(\vec{x}) \, || = r$.

(b) Fix an $r_o \in (0,1)$ and a $\theta_o \in (-\pi/2, \pi/2)$, and consider the path $\gamma : (0, 2\pi) \to \mathbb{R}^3$ defined by

$$\gamma(\xi) = T(\, (r_o, \xi, \theta_o) \,), \quad 0 < \xi < 2\pi.$$

Describe the image of this path.

(c) Fix an $r_o \in (0,1)$ and a $\xi_o \in (0, 2\pi)$, and consider the path $\gamma : (-\pi/2, \pi/2) \to \mathbb{R}^3$ defined by

$$\gamma(\theta) = T(\, (r_o, \xi_o, \theta) \,), \quad -\pi/2 < \theta < \pi/2.$$

Describe the image of this path.

(d) We accept the fact that T is a C^1-function. Write down the Jacobian matrix $(JT)(\vec{x})$ at a point $\vec{x} = (r, \xi, \theta) \in P$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function (that is, f is differentiable and $f' : \mathbb{R} \to \mathbb{R}$ is continuous). Let $F : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by by the formula $F(s,t) := f(st), \ \forall \, (s,t) \in \mathbb{R}^2$. Verify that $F \in C^1(\mathbb{R}^2, \mathbb{R})$, and prove that

$$s \, (\partial_1 F)(s,t) = t \, (\partial_2 F)(s,t), \quad \forall \, (s,t) \in \mathbb{R}^2.$$

Problem 3. Let $I \subseteq \mathbb{R}$ be an open interval and let $\gamma : I \to \mathbb{R}^n$ be a C^1-path with the property that

$$|| \, \gamma(t) \, || \ = \ 1, \quad \forall \, t \in I.$$

Fix a point $t_o \in I$, denote $\gamma(t_o) =: \vec{a}$ and $\gamma'(t_o) =: \vec{v}$. Prove that $\vec{v}$ is orthogonal to $\vec{a}$ (that is, prove that $\langle \, \vec{a}, \vec{v} \, \rangle = 0$).

In Problems 4 – 6 we examine the issue of points of local minimum/maximum for $f \in C^1(A, \mathbb{R})$, but where the local minimization or maximization is constrained to a specified subset S of A.

Definition. Let $A \subseteq \mathbb{R}^n$ be open, and let S be a non-empty subset of A. Let f be a function in $C^1(A, \mathbb{R})$ and let $\vec{a}$ be a point in S.

- If there exists $r > 0$ such that $f(\vec{x}) \leq f(\vec{a})$ for every $\vec{x} \in S \cap B(\vec{a}; r)$, then we say that $\vec{a}$ is is a *local maximum* for $f \mid S$.
- If there exists $r > 0$ such that $f(\vec{x}) \geq f(\vec{a})$ for every $\vec{x} \in S \cap B(\vec{a}; r)$, then we say that $\vec{a}$ is is a *local minimum* for $f \mid S$.
- If $\vec{a}$ is either a local maximum or a local minimum for $f \mid S$, then we say that $\vec{a}$ is a *local extremum* point for $f \mid S$.

In connection to the above definition, we can define a concept of "tangent vector to S", as follows.

Definition. Let S be a non-empty subset of $\mathbb{R}^n$ and let $\vec{a}$ be a point of S. A vector $\vec{v} \in \mathbb{R}^n$ is said to be *tangent* to S at $\vec{a}$ if there exists a C^1-path $\gamma : (-1, 1) \to \mathbb{R}^n$ such that $\gamma(t) \in S$ for every $t \in (-1, 1)$, such that $\gamma(0) = \vec{a}$, and such that $\gamma'(0) = \vec{v}$.

Problem 4. Let A be an open subset of $\mathbb{R}^n$ and let S be a non-empty subset of A. Let f be a function in $C^1(A, \mathbb{R})$. Let $\vec{a}$ be a point in S, and suppose that $\vec{a}$ is a local extremum point for $f \mid S$. Prove that the gradient vector $(\nabla f)(\vec{a})$ is perpendicular to every vector $\vec{v}$ which is tangent to S at $\vec{a}$.

Problem 5. Let A be an open subset of $\mathbb{R}^n$ and let f be a function in $C^1(A, \mathbb{R})$. Let us fix a value $\alpha \in \mathbb{R}$ which is taken by f, and let us put

$$S := \{\vec{x} \in A \mid f(\vec{x}) = \alpha\}$$

(this set S is called the *level set* of f, corresponding to the value α). We will assume that S has the following property:

$$(*) \qquad \left\{ \begin{array}{l} \text{For every } \vec{a} \in S, \text{ the linear span of the vectors tangent} \\ \text{to } S \text{ at } \vec{a} \text{ is a linear subspace of dimension } n-1 \text{ in } \mathbb{R}^n. \end{array} \right.$$

Now suppose that g is another function in $C^1(A, \mathbb{R})$, and that $\vec{a} \in S$ is a point of local extremum for $g \mid S$. By using the result in Problem 4, prove that the gradient vectors $(\nabla f)(\vec{a})$ and $(\nabla g)(\vec{a})$ are colinear.

The colinearity shown in Problem 5 is a trick used in extremum problems, which is referred to under the name of "Lagrange multipliers". The next problem is an illustration of how this works. (We accept that the condition $(*)$ from Problem 5 is satisfied by the sphere, hence that the trick of the Lagrange multipliers can be indeed applied.)

Problem 6. Let S be the unit sphere in $\mathbb{R}^3$, $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. By using Lagrange multipliers, determine the biggest and the smallest possible values of the expression $3x - yz$ when $(x, y, z) \in S$.