Math 249
Assignment 1

Due: Wednesday, January 19

1. (5 points) Use induction on n and the recurrence

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k-1}$$

   to prove the binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

2. The sets $S_1, \dots, S_m$ form a *partition* of S if their union is S and $S_i \cap S_j = \emptyset$ when $i \neq j$.
   (a) (3 points) If $n = 2m$, prove that the number of partitions of n with two parts of size m is $\binom{2m-1}{m-1}$.
   (b) (3 points) If $n = 2m$, prove that the number of partitions of n with all parts of size two is

$$\frac{(2m)!}{2^m m!} = (2m-1)(2m-3)\cdots 1.$$

3. (5 points) Construct a bijection between the even and odd subsets of $\{1, \dots, n\}$, and hence deduce that if $n > 1$,

$$\sum_{k \geq 0} \binom{n}{2k} = \sum_{k \geq 0} \binom{n}{2k+1}.$$

   (Here we take the view that $\binom{n}{k} = 0$ if $n < k$.)

4. (4 points) Prove the identity in the last question by applying the binomial theorem and noting that $(1-1)^n = 0$ (when $n > 0$).

5. (5 points) Let $\operatorname{sur}(n,k)$ denote the number of functions from an n-element set to a k-element set that are surjections. Prove that

$$m^n = \sum_{k=1}^{m} \binom{m}{k} \operatorname{sur}(n,k).$$