Math 249
Assignment 10

Due: Wednesday, March 30

1. Prove that a connected 4-regular graph has no cut-edge (that is, an edge whose deletion disconnects the graph).

2. By finding a subdivision of $K_{3,3}$, prove that the 4-cube is not planar.

3. It is known that if can draw a graph on the torus such that no edges cross and each face is a 2-cell, then
$$v - e + f = 0.$$
Using this, find an upper bound on the average degree of a simple graph which has such a drawing.

4. Suppose we have a group of people such that any two different people have exactly one friend in common. (We assume friendship is a symmetric and irreflexive relation.) Following the steps below, show that such a group must contain a politician—someone who is everyone's friend.

   Define a graph G with the people as vertices, and with two people adjacent if they are friends. Let A be the adjacency matrix of G.

   (a) Show that if there is no politician, then G is regular.

   (b) If G is regular of degree k, prove that $A^2 - (k-1)I = J$.

   (c) By computing eigenvalues and multiplicities, show that if G is regular then $k = 2$ and $G = K_3$.

   BonusBonusBonus

5. (Bonus points) Let G be a graph such that each pair of vertices has an odd number of common neighbors. Prove that $|V(G)|$ is odd.