Math 249
Assignment 2

Due: Wednesday, January 26

1. Suppose we have a circle with $2n$ distinct points marked on it. Determine the number of ways these $2n$ points can be joined by n chords, such that each point is on a chord. (Induction will do it.) Then determine a recurrence for the numbers of ways we can join the $2n$ points by n chords that do not cross and and cover each point on the circle.

2. By considering the coefficients in the series

$$(1+x)^a(1+x)^b,$$

   prove the Chu-Vandermonde identity

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}.$$

3. In the lectures we derived a bijection from permutations of $\{1,\ldots,n\}$ to the Cartesian product of sets. Using this, determine the generating series for the number of permutations of $\{1,\ldots,n\}$ weighted by the number of inversions.

4. Let q be a variable. Define $[n]$ by

$$[n] := \frac{q^n - 1}{q-1},$$

   and define $[n]!$ by $[0]! = 1$ and

$$[n+1]! = [n+1][n]!$$

   and define

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

   (Note that $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$. We call $[n]!$ the q*-factorial function* and $\begin{bmatrix} n \\ k \end{bmatrix}$ the q*-binomial coefficient*.) Prove that

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

5. Show that

$$[n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!$$

   and using this express $\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}}$ as a power of q times $\begin{bmatrix} n \\ k \end{bmatrix}$. Finally derive a second recurrence for $\begin{bmatrix} n \\ k \end{bmatrix}$, analogous to the one in the previous exercise.