Math 249
Assignment 3

Due: Wednesday, February 2

1. Let S be the set of all subsets of $\{0,\ldots,n-1\}$ and define the weight of a subset to be the sum of its elements. Let T be the subset of S consisting of the subsets that do not contain $n-1$. Let $\Phi_S(x)$ and $\Phi_T(x)$ be the generating series for S and T respectively.
   (a) (3 points) Compute $\Phi_S(x)$ when $n=4$.
   (b) (4 points) Prove that $\Phi_S(x) = (1+x^{n-1})\Phi_T(x)$.
   (c) (2 points) Give an expression for $\Phi_S(x)$ as a product of n polynomials.

2. Let $A(x)$ and $B(x)$ be series with constant terms equal to 1 such that $B(x) = A(x)^2$. Note that if $C(x)$ is the series $\sum_{n\geq 0} c_n x^n$, then its derivative $C'(X)$ is $\sum_{n\geq 0}(n+1)c_{n+1}x^n$.
   (a) (2 points) Using the usual rules from Calculus, show that $A(x)B'(x) = 2A'(x)B(x)$.
   (b) (2 points) Assuming that $a_k = \langle x^k, A(x)\rangle$ and $b_k = \langle x^k, B(x)\rangle$, compute formulas for the coefficient of x^n in $A(x)B'(x)$ and in $A'(x)B(x)$.
   (c) (5 points) Apply the previous material with $B(x) = (1-x)^{-1}$ to compute the first five coefficients of $(1-x)^{-1/2}$.

3. We say that two words α and β in $(a+b)^*$ are *conjugate* if there are words γ_1 and γ_2 such that
$$\alpha = \gamma_1\gamma_2, \quad \beta = \gamma_2\gamma_1.$$
   Equivalently β is conjugate to α if it is a cyclic shift of α. Let L denote the set of words α in $(a+b)^*$ such that the number of a's is one plus the number of b's and, in any prefix of α, the number of a's greater than the number of b's.
   (a) Prove that if α and β are conjugate words in L, then $\alpha = \beta$.
   (b) Prove that if α has length n, then it has n distinct conjugates.
   (c) Show that if β is a word over $\{a,b\}$ that contains $n+1$ a's and n b's, then β is conjugate to a unique word in L.
   (d) Determine the number of words in L with length $2n+1$.
   (e) Determine the number of Catalan paths.