Math 249
Assignment 1

Due: Wednesday, January 19

1. (5 points) Use induction on n and the recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

to prove the binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Solution: Assume that

$$(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n-1}{k} x^k.$$

Then

$$\begin{aligned}
(1+x)^n &= (1+x)\sum_{k=0}^{n} \binom{n-1}{k} x^k \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + \sum_{k=1}^{n} \binom{n-1}{k-1} x^k \\
&= 1 + \sum_{k=1}^{n-1} \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) x^k + x^n \\
&= 1 + \sum_{k=1}^{n-1} \binom{n}{k} + x^n,
\end{aligned}$$

where we used our recurrence in the last line. Since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$, the result follows at once.

2. The sets $S_1, \dots, S_m$ form a *partition* of S if their union is S and $S_i \cap S_j = \emptyset$ when $i \neq j$.

   (a) (3 points) If $n = 2m$, prove that the number of partitions of $\{1, \dots, n\}$ with two parts of size m is $\binom{2m-1}{m-1}$.

   (b) (3 points) If $n = 2m$, prove that the number of partitions of $\{1, \dots, n\}$ with all parts of size two is

$$\frac{(2m)!}{2^m m!} = (2m-1)(2m-3)\cdots 1.$$

Solution:

(a) Suppose π is a partition of $\{1,\dots,n\}$ with two parts of size m. Note that if we know the $m-1$ elements other than n in the part of π that contains n, then we can write down both parts of π. In other words we have a bijection from the set of partitions of $\{1,\dots,n\}$ with with two parts of size m to the set of subsets of $\{1,\dots,2m-1\}$ with size $m-1$. So the number of partitions is $\binom{2m-1}{m-1}$.

(b) We prove the result by induction on m. It is clearly true if $m=1$. Assume $m>1$ and assume inductively that the result holds for $m-1$. Suppose π is a partition of $\{1,\dots,2m\}$ and suppose that in π the element $2m$ is paired with j. If we delete the pair $\{j,2m\}$ from π, what is left over is a partition of the set

$$\{1,\dots,2m-1\}\setminus\{j\}$$

into $m-1$ pairs. By induction we find that the number of partitions of $\{1,\dots,2m\}$ into pairs where $2m$ is paired with j is equal to

$$(2(m-1)-1)(2(m-2)-1)\cdots 1 = (2m-3)(2m-5)\cdots 1$$

Since this does not depend on j and since we have $2m-1$ possible values for j, we conclude that the number of partitions of $\{1,\dots,2m\}$ into pairs is as stated.

3. (5 points) Construct a bijection between the even and odd subsets of $\{1,\dots,n\}$, and hence deduce that if $n>1$,

$$\sum_{k\ge 0}\binom{n}{2k} = \sum_{k\ge 0}\binom{n}{2k+1}.$$

(Here we take the view that $\binom{n}{\ell}=0$ if $n<\ell$.)

Solution: Let $\mathscr{O}$ denote the set of odd subsets of $\{1,\dots,n\}$. We define a function f on $\mathscr{O}$ as follows. If $\alpha\in\mathscr{O}$ and $1\in\alpha$, then

$$f(\alpha)=\alpha\setminus\{1\}$$

and if $1\notin\alpha$ then

$$f(\alpha)=\alpha\cup\{1\}.$$

If $\mathscr{E}$ is the set of even subsets of $\{1,\dots,n\}$, then f is defined on each element of $\mathscr{O}$ and it maps $\mathscr{O}$ to $\mathscr{E}$. We must show that f is a bijection.

Suppose α and β are distinct odd subsets. If $1\in\alpha$ and $1\notin\beta$ then $f(\alpha)\ne f(\beta)$ and similarly if $1\notin\alpha$ and $1\in\beta$ the $f(\alpha)\ne f(\beta)$. If 1 lies in both α and β then since $\alpha\ne\beta$ we have

$$f(\alpha)=\alpha\setminus\{1\}\ne\beta\setminus\{1\}=f(\beta).$$

If neither α nor β contains 1, then $\alpha \cup 1 \neq \beta \cup 1$. We conclude that f is injective.

We now show f is surjective. If γ is an even subset and $1 \notin \gamma$ then $\gamma = f(\gamma \cup 1)$; if $1 \in \gamma$ then $\gamma = f(\gamma \setminus 1)$. Hence f is surjective and therefore it is a bijection.

4. (4 points) Prove the identity in the last question by applying the binomial theorem and noting that $(1-1)^n = 0$ (when $n > 0$).

Solution: We have

$$\begin{aligned}(1-x)^n &= \sum_{k=0}^{n} \binom{n}{k} (-x)^k \\ &= \sum_{\ell=0} \binom{n}{2\ell} (-x)^{2\ell} + \sum_{\ell=0} \binom{n}{2\ell+1} (-x)^{2\ell+1} \\ &= \sum_{\ell=0} \binom{n}{2\ell} x^{2\ell} - \sum_{\ell=0} \binom{n}{2\ell+1} x^{2\ell+1}\end{aligned}$$

If we set $x = 1$ in this, our identity follows.

5. (5 points) Let $\text{sur}(n,k)$ denote the number of functions from an n-element set to a k-element set that are surjections. Prove that

$$m^n = \sum_{k=1}^{m} \binom{m}{k} \text{sur}(n,k).$$

Solution: We note that m^n is the number of functions from an n element set (say N) to an m-element set (say M). The range of such a function is a non-empty subset of M. If $M_1 \subseteq M$ and $|M_1| = k$, the number of functions whose range is M_1 is equal to $\text{sur}(n,k)$. Since M has $\binom{m}{k}$ such sets of size k, the number of functions whose range has size k is

$$\binom{m}{k} \text{sur}(n,k)$$

and therefore

$$m^n = \sum_{k=1}^{m} \binom{m}{k} \text{sur}(n,k).$$