PM 345 Ring Theory, Assignment 2 Due Wed Oct 26

**1:** (a) Find all the units and all the zero divisors in the ring $\mathbf{Z}_4[i]$.

(b) Without proof, list all of the subrings of $\mathbf{Z}_4[i]$.

(c) Without proof, list all of the ideals in $\mathbf{Z}_4[i]$.

(d) Find all primes $p$ with $p < 20$ such that $\mathbf{Z}_p[i]$ is a field.

**2:** (a) Find every ring homomorphism $\phi : \mathbf{Q}[\sqrt{2}] \to \mathbf{Q}[\sqrt{2}]$.

(b) Find every ring homomorphism $\phi : \mathbf{Z}_6 \to \mathbf{Z}_{10}$.

(c) Find every ring homomorphism $\phi : M_2(\mathbf{Z}) \to \mathbf{Z}$.

(d) Find the number of ring homomorphisms $\phi : \mathbf{Z} \to M_2(\mathbf{Z}_p)$, where $p$ is prime.

**3:** (a) Determine whether $\mathbf{Q}[\sqrt{2}] \cong \mathbf{Q}[\sqrt{3}]$.

(b) Determine whether $\mathbf{Z}_5[i] / \langle 2+i \rangle \cong \mathbf{Z}_5$.

(c) Determine whether $U_n(R) \cong L_n(R)$ where $R$ is any ring and $U_n(R)$ and $L_n(R)$ are the rings of upper- and lower-triangular matrices with entries in $R$.

(d) Determine whether $\mathbf{Z}_p[x] / \langle x^2 - 1 \rangle \cong {\mathbf{Z}_p}^2$, where $p$ is prime.

**4:** (a) Given $a, b \in \mathbf{Z}$, find the number of elements in $\mathbf{Z}^2 / \langle (a,b) \rangle$.

(b) Given $a, b \in \mathbf{Z}$, find the number of elements in $\mathbf{Z}[i] / \langle a + bi \rangle$.

(c) Given $n, k \in \mathbf{Z}^+$, find the number of elements in $\mathbf{Z}_n[x] / \langle x^k + 1 \rangle$.

(d) Given $A = \begin{pmatrix} 12 & 24 \\ 20 & 30 \end{pmatrix}$, find the number of elements in $M_2(\mathbf{Z}) / \langle A \rangle$.

**5:** Let $F$ be a field. Consider the ring $F[x,y]$.

(a) Show that $\langle x+y, x^2+y \rangle = \langle x, y \rangle \langle x-1, y+1 \rangle$.

(b) Show that for all $a, b \in F$, we have $\langle x-a, y-b \rangle = \{ f \in F[x,y] \,|\, f(a,b) = 0 \}$.

(c) Show that for all $a, b \in F$, the ideal $\langle x-a, y-b \rangle$ is maximal.

(d) Show that $F$ is finite if and only if the ring homomorphism $\phi : F[x,y] \to \mathrm{Func}(F^2, F)$ given by $\phi(f) = f$ is surjective.