PM 345 Ring Theory, Assignment 3 — Due Fri Nov 18

1: Let R be a unique factorization domain. Prove each of the following statements.

(a) Every irreducible element in R is prime.

(b) For all $a_1, a_2, a_3, \cdots \in \mathbf{R}$ with $\langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq \langle a_3 \rangle \subseteq \cdots$, there exists $n \geq 1$ such that $\langle a_k \rangle = \langle a_n \rangle$ for all $k \geq n$.

(c) Every non-zero prime ideal in R contains a non-zero prime principal ideal.

(d) If for all $a, b \in R$ there exists $d \in R$ with $\langle d \rangle = \langle a, b \rangle$, then R is a principal ideal domain.

2: Consider the ring $\mathcal{C}^0(\mathbf{R})$ of continuous functions $f : \mathbf{R} \to \mathbf{R}$. Prove each of the following.

(a) The units in $\mathcal{C}^0(\mathbf{R})$ are the nowhere zero functions, and the zero-divisors in $\mathcal{C}^0(\mathbf{R})$ are the functions which are not identically zero, but which are zero in some open interval.

(b) There are no irreducible elements and no prime elements in $\mathcal{C}^0(\mathbf{R})$.

(c) There exists an infinite ascending chain $\langle f_1 \rangle \subsetneqq \langle f_2 \rangle \subsetneqq \cdots$ of principal ideals in $\mathcal{C}^0(\mathbf{R})$

(d) There exist $f, g \in \mathcal{C}^0(\mathbf{R})$ such that $f \sim g$ but $f \neq gu$ for any unit u.

3: Consider the ring $F\big[[x]\big]$ of formal power series in x, where F is a field.

(a) Find the units in $F\big[[x]\big]$.

(b) Find the equivalence classes under association in $F\big[[x]\big]$.

(c) Find the irreducible elements in $F\big[[x]\big]$.

(d) Show that $F\big[[x]\big]$ is a Euclidean domain.

4: (a) List all of the irreducible elements $z \in \mathbf{Z}[\sqrt{6}\,i]$ with $||z|| \leq 10$.

(b) List all of the elements $z \in \mathbf{Z}[\sqrt{6}\,i]$ with $||z|| \leq 10$ which do not factor uniquely.

(c) Show that $\mathbf{Z}\left[\frac{1+\sqrt{11}\,i}{2}\right]$ is a Euclidean domain under the norm $N(z) = ||z||^2$.

(d) Show that $\mathbf{Z}\left[\frac{1+\sqrt{43}\,i}{2}\right]$ is not a Euclidean domain under any norm.

5: For a positive integer d which is not a square, in the ring $\mathbf{Z}[\sqrt{d}]$ and in the field $\mathbf{Q}[\sqrt{d}]$, consider the norm given by $N(x + y\sqrt{d}) = \left|x^2 - dy^2\right|$.

(a) Show that for all $z \in \mathbf{Q}[\sqrt{d}]$ we have $N(z) = 0 \iff z = 0$, and for all $z, w \in \mathbf{Q}[\sqrt{d}]$ we have $N(zw) = N(z)N(w)$, and for all $z \in \mathbf{Z}[\sqrt{d}]$ we have $N(z) = 1 \iff z$ is a unit.

(b) Show that $\mathbf{Z}[\sqrt{3}]$ is a Euclidean domain under this norm.

(c) Show that $\mathbf{Z}[\sqrt{5}]$ is not a unique factorization domain.

(d) Show that $\mathbf{Z}[\sqrt{3}]$ and $\mathbf{Z}[\sqrt{5}]$ each have infinitely many units.