PM 345 Ring Theory, Assignment 4 Due Fri Dec 2

1: Factor each of the following polynomials into irreducibles.

(a) $f(x) = 12x^5 + 30x^4 - 6x^3 + 42x^2 + 12x - 18 \in \mathbf{Z}[x]$

(b) $f(x) = x^4 - x^3 + 6x^2 + 2x + 12 \in \mathbf{Z}[\sqrt{2}\,i][x]$

(c) $f(x) = x^{12} - 1 \in \mathbf{R}[x]$

(d) $f(x) = x^3 - 3x + 1 \in \mathbf{R}[x]$ (hint: let $x = z - z^{-1}$ with $z \in \mathbf{C}$)

2: (a) Show that $f(x) = x^5 + 6x^4 + 3x^3 - x^2 + 5x + 1$ is irreducible in $\mathbf{Z}[x]$.

(b) Show that $f(x) = x^4 + (1-2i)x^3 + (3+4i)x + 5$ is irreducible in $\mathbf{Z}[i][x]$.

(c) Show that $f(x) = (x-1)(x-2)(x-3)\cdots(x-n) - 1$ is irreducible in $\mathbf{Z}[x]$.

(d) Show that $f(x) = x^4 + 1$ is reducible in $\mathbf{Z}_p[x]$ for all proimes $p < 20$.

3: (a) Let $f(x) = x^5 + 2x^4 + 2x^3 + x^2 + x + 2$ and $g(x) = x^3 + x^2 + 2x + 2$ in $\mathbf{Z}_3[x]$. Find the monic polynomial $d \in \mathbf{Z}_3[x]$ such that $d = \gcd(f, g)$ and then find monic polynomials $u, v \in \mathbf{Z}_3[x]$ such that $fu + g\,v = d$.

(b) List all the irreducible monic polynomials of degree 2 in $\mathbf{Z}_3[x]$.

(c) Find the number of irreducible polynomials of degree 3 in $\mathbf{Z}_3[x]$.

(d) Let $f(x) = x^3(x+1)^2(x^2+1) \in \mathbf{Z}_3[x]$. Find the number of distinct ideals in $\mathbf{Z}_3[x]/\langle f\rangle$.

4: (a) Show that $f(x) = x^2 + 3x + 2$ is irreducible in $\mathbf{Z}\big[[x]\big]$ but reducible in $\mathbf{Z}[x]$.

(b) Find a polynomial $f \in \mathbf{Z}[x]$ such that f is irreducible in $\mathbf{Z}[x]$ but reducible in $\mathbf{Z}\big[[x]\big]$.

(c) Find a ring R with $\mathbf{Z} \subset R \subset \mathbf{Z}[x]$ (where the inclusions are subrings) such that $x^2 - 1 \in R$ and $x^3 - 1 \in R$ and $x^2 - 1$ and $x^3 - 1$ are both irreducible in R.

(d) Find an integral domain R, with $\mathbf{Z}[x] \subset R$ (as a subring), such that $x+1$ and $x-1$ are both reducible in R (to be completely rigorous, you will not have $\mathbf{Z}[x] \subset R$ but you will have an injective homomorphism $\phi : \mathbf{Z}[x] \to R$, and you can identify $\mathbf{Z}[x]$ with $\phi(\mathbf{Z}[x])$).

5: In parts (a), (b) and (c), let R be a commutative ring with $1 \neq 0$.

(a) Show that if there is an irreducible element $a \in R$ then $R[x]$ is not a PID.

(b) Show that if $f = \sum_{i=0}^{n} c_i x^i$ is a zero-divisor in $R[x]$ then there exists $a \in R$ such that for all $i = 0, 1, \cdots, n$ we have $a\,c_i = 0$.

(c) Show that for $f = \sum_{i=0}^{n} c_i x^i \in R[x]$, f is a unit in $R[x]$ if and only if c_0 is a unit in R and $c_1, c_2, \cdots, c_n$ are idempotents in R (x is an *idempotent* when $x^n = 0$ for some $n \in \mathbf{Z}^+$).

(d) Find the number of units of degree 1 in $\mathbf{Z}_n[x]$ in terms of the prime factorization of n.