PM 346
Assignment 1 - Solutions

1. If $a \in G - \{1\}$ then a has order p, q or pq. If $o(a) = pq$ we are done so assume WLOG that $o(a) = p$. The subgroup $\langle a \rangle$ is normal in G so $|G/\langle a\rangle| = q$. Choose $b \in G$ so that $o(\overline{b}) = q$ (in $G/\langle a \rangle$). Once again WLOG $o(b) = q$, so $(ab)^p = a^p b^p = b^p \neq 1$, $(ab)^q = a^q \neq 1$
$\therefore$ $o(ab) = pq$ and we are done

2. If $a, b \in G$, $o(ab) = 2$, so $(ab)^{-1} = ab$. ie $\cancel{b^{-1}}$ $ba = b^{-1}a^{-1} = ab$.
$\therefore$ G is abelian

3. Define a relation on G by $a \sim b$ if $a = b$ or $a^{-1} = b$. This is an equivalence relation. If $o(c) \neq 1$ or 2, then $|[c]| = 2$ (ie $c \neq c^{-1}$). As G is a union of equivalence classes and $|[1]| = 1$, there must be some $c \neq 1$ st. $|[c]| = 1$ (since $|G|$ is even). $\therefore$ $o(c) = 2$.

4. Suppose that Q_2 is generated by $\dfrac{a}{2^n}$. Then $\dfrac{1}{2^{n+1}} = n\left(\dfrac{a}{2^n}\right) - a$

contradiction $\therefore$ $\mathbb{Q}_2$ is <u>not</u> cyclic.

If H is a proper subgroup then there is an upper bound for n s.t. $\frac{a}{2^n} \in H$ (a odd) (otherwise $\frac{1}{2^n} \in H$ for all n and $H = \mathbb{Q}_2$).

Let k be the least upper bound. $\therefore$ $\frac{a}{2^k} \in H$, a odd and as $H \supseteq \mathbb{Z}$, and $\gcd(a, 2^k) = 1$, we have $c, d \in \mathbb{Z}$ s.t. $ca + d2^k = 1$.

$$\therefore \frac{ca}{2^k} + d = \frac{ca + 2^k d}{2^k} = \frac{1}{2^k}$$

which generates H.

5 $a \sim b$ if $ab^{-1} \in H$

1) $a \sim a \Rightarrow 1 = aa^{-1} \in H$

2) if $a \in H$, then $a1^{-1} \in H$ so $a \sim 1$. By symmetry $1 \sim a$ $\therefore$ $a^{-1} = 1a^{-1} \in H$

3) If $a, b \in H$, then $a \sim 1$ and as $b^{-1} \in H$, $1 \sim b^{-1}$. By transitivity $a \sim b^{-1}$ ie $a(b^{-1})^{-1} \in H$ - so $ab \in H$.

6. The elements of S_3 can be expressed as $\{1, b, b^2, a, ab, ab^2\}$ where $b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Then $b^3 = 1 = a^2$ and $ba = ab^2$

The normal subgroups of S_3 are $\langle 1 \rangle$, $\langle b \rangle$ and S_3 (proved in class)

Any Cartesian product of normal subgroups of S_3 is normal in $S_3 \times S_3$. This gives 9 normal subgroups of $S_3 \times S_3$.

There is a homomorphism $\varphi : S_3 \times S_3 \longrightarrow C_2 \times C_2$ which sends $(a^i b^j, a^k b^l)$ to (a^i, a^k)

If N is a normal subgroup of $S_3 \times S_3$, then $\varphi(N)$ is a (normal) subgroup of $C_2 \times C_2$ – which has 5 subgroups

<u>Case 1</u> $\varphi(N) = \langle (a,1) \rangle$ (or $\langle (1,a) \rangle$)

$\therefore x = (ab^i, b^j) \in N$ for some i, j.

Conjugating by $(a,1)$ (or $(1,a)$) we get $y = (ab^{2i}, b^j) \in N$ $((ab^i, b^{2j}) \in N)$

$yx^{-1} = (b^i, 1)$ (or $(1, b^j)$)

$\therefore$ we pick up all elements in $S_3 \times \langle b^j \rangle$

$\therefore$ $N = S_3 \times \{1\}$ or $S_3 \times B$

where $B = \langle b \rangle$.

Case 2 $\varphi(N) = \langle (a,a) \rangle$.

So $x = (ab^i, ab^j) \in N$.

Conjugating by $(b,1)$ $(a\ (1,b))$ we get $y = (ab^{i+2}, ab^j)$ $(a\ (ab^i, ab^{j+2})$.

Then $yx^{-1} = (b^2, 1)$ $(a\ (1, b^2)$

& with a bit of work we see that

$N = \{ (a^m b^i, a^m b^j) : \quad m = 0, 1,\ i, j = 0, 1, 2 \}$.

Case 3 $\varphi(N) = C_2 \times C_2$.

From Cas 1 we see that $S_3 \times \{1\} \leq N$ and $\{1\} \times S_3 \leq N$, so $N = S_3 \times S_3$

Case 4 $\varphi(N) = \langle (1,1) \rangle$.

Suppose N is not one of $\{1\} \times \{1\}$ $B \times \{1\}$ or $\{1\} \times B$.

$\therefore \exists (b^i, b^j) \in N$, $i, j \neq 0$.

Conjugate by $(a, 1)$ to get (b^{2i}, b^j)

and then we get $(b^i, 1) \in N$.

Similarly $(1, b^j) \in N$. So $N = B \times B$