PM 346

Assignment 2 Solutions

1.

Suppose that $C_6 = \langle g \rangle$. A homomorphism from C_6 to S_3 will be given by specifying where g goes to. All we need is that $o(\alpha(g)) \mid o(g) = 6$ $\therefore$ We can send g to any element of S_3 $\therefore$ There are 6 homomorphisms.

2. The image of S_3 in C_6 has to be cyclic. and S_3 has only one proper nontrivial normal subgroup $(\neq S_3)$ namely the 3-element subgroup H. Then $S_3/H \cong C_2$. There is one subgroup of C_6 of order 2 and we can map $S_3 \to S_3/H \to$ group of order 2 in C_6 & we can map every element of S_3 to 1. There are 2 homomorphisms.

3. Suppose that $\alpha \in Aut(\mathbb{Q},+)$ and suppose that $\alpha(b) = a$.

Then $\alpha(nb) = \alpha(\underbrace{b+b+\dots+b}_{n\text{-times}}) = n\alpha(b) = na$.
$n \in \mathbb{P}$

If $\alpha(1) = c$, $\alpha(n) = nc$, $\alpha(-n) = -nc$.
Also $c = \alpha(n \cdot \frac{1}{n}) = n\,\alpha(\frac{1}{n})$, so $\alpha(\frac{1}{n}) = \frac{1}{n}c$.

In general then $\alpha(\frac{m}{n}) = \frac{m}{n}c$.
Thus there is precisely one automorphism for each $c \in \mathbb{Q} - \{0\}$ and this automorphism satisfies $\alpha(1) = c$.

4. Suppose $a \in G - H$. Then
$H \dot{\cup} aH = G$ and $H \dot{\cup} Ha = G$.
disjoint union

It follows that $aH = Ha$ — so every left coset is a right coset $\therefore$ H is normal

5. a) If $a \in N, b \in H$ $\quad ab = b\underbrace{b^{-1}ab}_{\in N}$
$= ba'$, $a' \in N$
So $NH \subseteq HN$. Similarly $HN \subseteq NH$ so NH is a subgroup ($n_1h_1n_2h_2 = n_1n_2'h_1h_2$ (closed under mult) $(nh)^{-1} = h^{-1}n^{-1} = (n^{-1})'h^{-1}$ (closed under inverse)

b) If $c \in G$, $a \in N$, $b \in H$,
$c^{-1}abc = \underbrace{c^{-1}ac}_{\in N}\underbrace{c^{-1}bc}_{\in H} \in NH$.

So NH is closed under conjugation.

c) $1 \in N \cap H$. $a, b \in N \cap H \Rightarrow ab, a^{-1} \in N$ (since N is a subgroup) and $ab, a^{-1} \in H$ (since H is a subgroup) $\therefore$ $N \cap H$ is a subgroup

d) $N \cap H$ is a subgroup of G and therefore of H. If $h \in H$, $h^{-1}(N \cap H)h \subseteq N$ (since N is normal) and $\subseteq H$ since H is a subgroup $\therefore$ $N \cap H$ is closed under conjugation by elements of H.

6). $a \in H,\ b \in N \Rightarrow \underbrace{a^{-1}}_{\in H}\underbrace{b^{-1}ab}_{\in H} \in H$

and $\underbrace{a^{-1}b^{-1}a}_{\in N}\underbrace{b}_{\in N} \in N$, so $a^{-1}b^{-1}ab \in H \cap N = \{1\}$

Thus $a^{-1}b^{-1}ab = 1$ ie $ab = ba$.