PMath 346 Assignment 3 - Solutions

1 If $n=3$, $|H|=3$ and there is only one subgroup of S_3 of order 3

$\underline{n \geq 4}$ If H A_n, then

$$n! = |HA_n| = \frac{|H| \cdot |A_n|}{|H \cap A_n|} = \frac{\frac{1}{4}(n!)^2}{|H \cap A_n|}$$

$\therefore$ $|H \cap A_n| = \frac{1}{4} n! = \frac{1}{2} |A_n|$.
But $H \cap A_n$ is a normal subgroup of A_n which is impossible if $n \geq 5$.
If $n=4$, then we proved in class that A_4 has no subgroup of order 6.

2. Suppose $|G| = m$. Let $n = 2m$.
Send $g \in G$ to the permutation on $G \times G$ induced by left multiplication.
This is always even and embeds G into A_{2m}.
(Essentially we are embedding G into $G \times G$ by $g \mapsto (g,g)$ + then embedding g into A_m twice).

3. 1) The element has cycle structure

$[5,5]$ or $[5,1,1,1,1,1]$

The number of such elements is

$$\frac{10!}{2\times 5^2} + \frac{10\times 9\times 8\times 7\times 6}{5} = 6048 + 72576 = 78,624$$

ii) Cycle structure — number of elements

Cycle structure	number of elements
$[6,1,1,1,1]$	$\frac{10!}{6\times 4!} = 25,200$
$[3,3,2,2]$	$\frac{10!}{3^2\times 2^2\times 4} = 25,200$
$[3,3,2,1,1]$	$\frac{10!}{3^2\times 2\times 2\times 2} = 50,400$
$[3,2,2,1,1,1]$	$\frac{10!}{3\times 4\times 2\times 6} = 25,200$
$[3,2,2,2,1]$	$\frac{10!}{3\times 8\times 6} = 25,200$
$[3,2,1,1,1,1,1]$	$\frac{10!}{3\times 2\times 5!} = 5040$
$[6,3,1]$	$\frac{10!}{6\times 3} = 201,600$
$[6,2,2]$	$\frac{10!}{6\times 4\times 2} = 75,600$
$[6,2,1,1]$	$\frac{10!}{6\times 2\times 2} = 151,200$
	$584,640$

iii) $\prod i^{l_i} \times (l_i)!$ $l_i = \#$ i-cycles

$l_3 = 1$, $l_2 = 2$, $l_1 = 3$.

$\therefore |C_G[(123)(45)(67)]| = (3^1 \times 1!) \times (2^2 \times 2!) \times (1^3 \times 3!)$
$= 144$.

4. i) If $|G/Z(G)|$ is prime, then $G/Z(G)$ is cyclic, say generated by $\bar{g}$.

$\therefore$ every element of G can be expressed as $g^k a$, $a \in Z(G)$.

But if $b \in Z(G)$, $(g^k a)(g^l b)$
$= g^{k+l} ab = (g^l b)(g^k a)$, so G is abelian

ii) If $|G| = p^2$, G is a p-group so $Z(G) \neq \{1\}$ $\therefore |Z(G)| = p$ or p^2.
From i) we get $|Z(G)| = p^2$
ie G is abelian.

5. $|G| = 726 = 2 \times 3 \times 11^2$.
The abelian case is straightforward.
If $Z(G) \neq \{1\}$ we are done

By the class eqn

$$726 = 1 + \sum_{\substack{\text{one } a \\ \text{from each} \\ \text{conj cl.} > 1}} \frac{|G|}{|C_G(a)|}$$

Clearly 11 cannot divide all the numbers $\dfrac{|G|}{|C_G(a)|}$, so one of them

must be 2, 3 or 6. In this case we have a proper subgroup $C_G(a)$ s.t. $|G| \nmid [G : C_G(a)]!$

$\therefore$ G is not simple