PMath 346 Assignment 4 - Solutions

1. A_5 is simple so $n_p \neq 1$ for all primes p dividing $60 = 5 \times 3 \times 2^2$

$n_5 \neq 1$ & $n_5 \mid 12$ & $n_5 \equiv 1 \pmod 5 \Rightarrow n_5 = 6$

$n_3 \neq 1$ & $n_3 \mid 20$ & $n_3 \equiv 1 \pmod 3 \Rightarrow n_3 = 4$ or 10

$n_2 \mid 15$ & $n_2 \neq 1$ & $n_2 \equiv 1 \pmod 2 \Rightarrow n_2 = 3$ or 5

A_5 has $\frac{5 \times 4 \times 3}{3} = 20$ elements of order 3 $\therefore n_3 = 10$

A_5 has no elements of order 4 but has $\frac{5 \times 4 \times 3 \times 2}{8} = 15$ elements of order 2 (cycle structure $[2,2,1]$)

The Sylow-5-subgroups are $\cong C_5$

" " 3-subgroups are $\cong C_3$

" " 2 " " " $C_2 \times C_2$

$A_5 < (12)(34), (13)(24) > \cong C_2 \times C_2$

and they are all conjugate, there are $\binom{5}{4} = 5$ such groups

$\therefore n_2 = 5$

2. D_5 has an element of order 5 $\therefore n \geq 5$

A_5 $[(15)(24)]\,(12345)\,[(15)(24)]$

$= (54321)$, we let $a = (15)(24)$

and $b = (12345)$ and we have
$D_5 = \langle a, b : a^2 = b^5 = 1, aba = b^{-1} \rangle$
embedded into A_5.
$n = 5$.

3. i) $\begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+u & b+v+uc \\ 0 & 1 & b+w \\ 0 & 0 & 1 \end{pmatrix}$

and $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+u & v+aw+b \\ 0 & 1 & w+b \\ 0 & 0 & 1 \end{pmatrix}$

If $\begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} \in Z(G)$, then $uc = aw$
for all a, c. Let $a=1, c=0 \Rightarrow w=0$
Similary' $u = 0$

ii) Clearly G has order 8
$o\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2$ (if $a \neq 0$) and
this element is not central
$\therefore$ G is not abelian and
is not the quaternion group
So $G \cong D_4$.

4. Check that the set is closed under matrix multiplication

Matrix multiplication is associative and since the set is finite, it most be a group.

Each of the six elements

$$\pm\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

has order 4 eg

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

D_4 has 2 elements of order 4, so this group is not isomorphic to D_4.

5 <u>Soln</u> A group of order 66 has a subgroup H of order 11 (by Cauchy's Thm).

Thus there is a homomorphism

$$\alpha : G \to S_6 \text{ given by}$$

$$g \longmapsto (aH \to gaH).$$

As [illegible] $\nmid$ [illegible]$!$, α is not $1:1$ and as $\ker \alpha \subseteq H$, $\ker\alpha = H$.

Thus $H \lhd G$. Let $a \in G$ be an element of order 3. Then $\langle a \rangle H$ is a subgroup of G and one can see by counting that

$$|\langle a \rangle H| = 33$$

As $3 \nmid 10$ $(= 11-1)$, this group is cyclic.