PMath 346 Assignment 5 – Solutions

1. Let $a = (12)(34)$ and $b = (1234)$

Then $aba^{-1} = aba = (2143) = b^{-1}$

So $\langle a, b \rangle$ is a 2-subgroup of S_4 & it has order 8.

But $a^2 = 1$, $b^4 = 1$ and $aba = b^{-1}$

So $\langle a, b \rangle \cong D_4$.

2. The group of symmetries is D_{15}.

The number of molecules is

$$\frac{1}{30} \sum F(g), \text{ where } g \in D_{15}$$

and $F(g)$ is the number of 15-gons fixed by g.

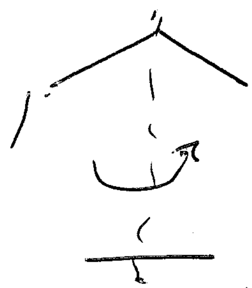
Case 1 $\quad g = 1 \qquad \# = t^{15}$

Case 2 $\quad g$ is a rotation of order 3 (there are 2 such elements) & t^5 fixed polygons in each case

Case 3 $\quad g$ is a rotation of order 5 (there are 4 such elements) & t^3 fixed polygons in each case

Case 4 $\quad g$ is a rotation of order 15 (there are 8 such elements) & t fixed polygons in each case

<u>Case 5</u> g is a reflection. There are 15 - reflections all of the same typ



+ there are t^8 fixed polygons in each case.

$\therefore$ # molecules $= \frac{1}{30}\left(t^{15} + 3t^5 + 4t^3 + 8t + 15t^8\right)$

3. If $h \in G - Z(G)$, then conjugation by h is a nontrivial automorphism

$\therefore$ WLOG G is abelian.

In an abelian group the map $x \longmapsto -x$ is an automorphism, so if this is the identity map then

$x = -x \quad \forall x \in G$

ie $2x = 0$ (writing additively).

$\therefore$ G is a vector space over $\mathbb{Z}_2$

$\therefore$ isomorphic to $C_2 \times C_2 \times \cdots \times C_2$

(or one can deduce this from the Structure Thm for abelian groups).

But if we have more than 1 copy of C_2 then there is an automorphism

$(a_1, a_2, a_3 \cdots, a_e) \mapsto (a_2, a_1, a_3 \cdots a_e)$

So there is at most one copy of C_2.

4. $144 = 2^4 \times 3^2$ The (abelian) groups of order 16 are

$C_{16}, C_8 \times C_2, C_4 \times C_4, C_4 \times C_2 \times C_2$

$C_2 \times C_2 \times C_2 \times C_2$. The groups of order 9 are C_9 and $C_3 \times C_3$.

$\therefore$ There are 10 isomorphism classes of groups of order 144 namely

group of order 16 $\times$ group of order 9.

5. a) C_9 has $\phi(9) = 6$ elements of order 9.

$\therefore$ There are $6 \times 3 \times 3 = 54$ elements of order 9.

b). C_4 has one element of order 2

and C_9, and C_3 each have 2 elements of order 3

$\therefore$ There are $1 \times$ # elements in $C_9 \times C_3 \times C_3$ of order 3

$= 1 \times (81 - 54 - 1) = 26$

elements of order 6

6. Q is the subgroup of the quaternion elements $\{\pm 1, \pm i, \pm j, \pm k\}$ under multiplication

As $-1 \in \langle -1 \rangle \cap \langle \pm i \rangle \cap \langle \pm j \rangle \cap \langle \pm k \rangle$ we see that any two of the seven. cyclic subgroups $\langle -1 \rangle, \langle \pm i \rangle, \langle \pm j \rangle, \langle \pm k \rangle$ intersect nontrivially. So if any two subgroups contain these cyclic subgroups, they intersect nontrivially.