PMath 346 Assignment 6 - Solutions

1 As $\mathrm{Aut}\, C_7 \cong C_6$ we can construct a semidirect product by finding an element of order 3 in $\mathrm{Aut}\, C_7$.

If $C_7 = \langle b \rangle$, then the automorphism α which sends b to b^2 has order 3 (α^3 sends b to $b^{2^3} = b^8 = b$).

Define $\overline{a}(b) = b^2$

$(b^k, a^l) * (b^m, a^n) = (b^k \overline{a}^l(b^m), a^{l+n})$
$= (b^{k+2^l m}, a^{l+n})$

The Sylow 7-subgroup is normal so there are 6 elements of order 7

Every other nonidentity element has order 3 so there are $21-(6+1)=14$ elements of order 3.

2. a) The order of $\mathrm{Aut}(C_p \times C_p)$ is the order of invertible matrices in $M_2(\mathbb{Z}_p)$.

This is because $C_p \times C_p$ is a $\mathbb{Z}_p$-vector space and an automorphism of the group is an automorphism of the vector space.

There are p^2-1 choices for the first row of the matrix. We do not want the second row to be a scalar multiple of the first row, so there are p^2-p choices for the second row.

The order of $\mathrm{Aut}(C_p \times C_p)$ is $(p^2-1)(p^2-p)$

b) If m is not square-free, then for some prime p, $p^2 \mid m$. Let $k = \frac{m}{p^2}$. Now $3 \mid (p^2-1)(p^2-p)$, so let α be an automorphism of $Aut(C_p \times C_p)$ of order 3.

Define $\underline{\quad} : C_3 \to Aut(C_k \times C_p \times C_p)$ by the generator a of C_3 is sent to the automorphism which sends (u, v, w) to $(u, \alpha(v, w))$

Construct the semidirect product extension of $C_k \times C_p \times C_p$ by C_3. This is nonabelian (since $\underline{\quad}$ is not the trivial map) and the order of the group is $3m$.

3. If $(ab), (bc) \in S_4$, then $(ab)(bc)(ab)^{-1}(bc)^{-1} = (ab)(ac) = (acb)$ is the commutator $[(ab),(bc)]$. Every 3-cycle is in the commutator subgroup, so $(S_4)' \supseteq A_4$. But S_4/A_4 is abelian so $S_4' = A_4$.

A_4 has a normal subgroup B of order 4 and no proper nontrivial subgroup of B is normal in A_4. A_4' is a normal subgroup of A_4 and A_4'/B is abelian, so $A_4' \subseteq B$. A_4 is not abelian, thus $A_4' \neq \{1\}$. Thus $A_4' = B$.

B is abelian and so $B' = \{1\}$.

We have proved that S_4 is solvable with soluable length 3.

4. (a) $b^m a b^{-m} = a b^{-2m}$

and $a \in \langle a \rangle \langle b^5 \rangle$, so $b^m \in N(\langle a \rangle \langle b^5 \rangle)$ if and only if $a b^{-2m} \in \langle a \rangle \langle b^5 \rangle$ ie. if and only if $5 \mid m$. $\therefore$

$\langle a \rangle \langle b^5 \rangle \leq N_{D_{10}}(\langle a \rangle \langle b^5 \rangle) \subseteq \langle a \rangle \langle b^5 \rangle$

so $N_{D_{10}}(\langle a \rangle \langle b^5 \rangle) = \langle a \rangle \langle b^5 \rangle$.

(b). $[a, b] = a b a^{-1} b^{-1} = b^{-2}$, so

$b^2 \in D_{10}'$. But $[D_{10} : \langle b^2 \rangle] = 4$, and $\langle b^2 \rangle \lhd D_{10}$, so $D_{10}/\langle b^2 \rangle$ is abelian and thus $D_{10}' \subseteq \langle b^2 \rangle$

Therefore $D_{10}' = \langle b^2 \rangle$.

4. Use induction on n.

If $n = 1$, the result is clear.

Otherwise G has an element g of order p in its centre, so $|G/\langle g \rangle| = p^{n-1}$.

Let $\alpha : G \longrightarrow G/\langle g \rangle$ be the natural map. If $m \leq n$, then $m-1 \leq n-1$ and by induction $G/\langle g\rangle$ has a subgroup H with $|H| = p^{m-1}$.

Then $\alpha^{-1}(H)$ is a subgroup of G of order p^m.

(H is a disjoint union of p^m cosets each of size p).