PMATH 351 Assignment #2

Due: Monday, October 17

1) Let d_1,d_2 and d_∞ be the metrics on $\mathbb{R}^n$ given by

$$d_1(x,y) = \sum_{i=1}^{n} \mid x_i - y_i \mid$$

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$d_\infty(x,y) = \max_{i=1,\ldots,n} \mid x_i - y_i \mid$$

Let τ_1,τ_2 and τ_∞ be the topologies induced by the above metrics.

Show that $\tau_1 = \tau_2 = \tau_\infty$.

2) For each of the following sets determine if it is open, closed or neither. Indicate the set of limit points, boundary points and interior points of each set.

a) $(0,1] \subset \mathbb{R}$.

n) $\mathbb{Q} \subset \mathbb{R}$.

c) $\mathcal{P}_1 = \{a_0 + a_1 x \mid a_i \in \mathbb{R}\} \subset (C[0,1], d_\infty)$.

d) $c_{00} = \{\{a_n\} \in l_\infty \mid a_n = 0$ for all but finitely many n's$\} \subset l_\infty$.

3) Least Upper Bound Property:

We say that α is an upper bound of $S \subset \mathbb{R}$ if $x \leq \alpha$ for all $x \in S$. We say that S is bounded above if it has an upper bound. We call α the *least upper bound* of S if α is an upper bound of S and if whenever β is an upper bound of S we have $\alpha \leq \beta$. We denote α by $lub(S)$ (We may define lower bounds and the *greatest lower bound* ($glb(S)$) in the obvious way) The *Least Upper Bound Property* states that every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound (or equivalently that every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound).

a) Prove the Monotone Convergence Theorem: Let $\{a_n\}$ be a sequence in $\mathbb{R}$ with $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges.

b) Prove the Nested Interval Theorem: Let $\{[a_n, b_n]\}$ be a sequence of closed intervals with $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for each $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

c) Show that the statement in Part b) may fail if we use open intervals.

d) Use the Nested Interval Theorem to show that if $S \subset \mathbb{R}$ is infinite and bounded, then it has a cluster point. (This is called the Bolzano-Weierstrass Theorem.)

e) Given a nonempty set $A \subset (X, d)$ we define the diameter of A to be $diam(A) = sup\{d(x, y) \mid x, y \in A\}$. Show that if A_n is a sequence of nonempty closed sets in $\mathbb{R}$ with $A_{n+1} \subseteq A_n$ and $diam(A_1) < \infty$, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

4) Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in $\mathbb{R}$ such that $[0, 1] \subset \bigcup_{\alpha \in I} U_\alpha$.

a) Show that there exists finitely many sets $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ such that $[0, 1] \subset \bigcup_{i=1}^{n} U_{\alpha_i}$.

b) Show that the statement in part a) can fail if we replace $[0, 1]$ with $(0, 1)$.

5) A map $\phi : (X, d_X) \to (Y, d_Y)$ is called an isometry if $d_Y(\phi(x_1), \phi(x_2)) = d_X(x_1, x_2)$.

a) Determine all possible isometries $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ and show that each such map is surjective.

b) Give an example of an isometry $\phi : (X, d_X) \to (X, d_X)$ that is not onto.

6) A topological space (X, τ) is called separable if there exists a countable subset $S \subset X$ such that $\overline{S} = X$.

Show that (l_1, d_1) is separable but (l_∞, d_∞) is not.

7) Let $\mathbf{x}_n = \{x_{n,1}, x_{n,2}, x_{n,3}, \ldots\} \in l_\infty$. Show that if $\mathbf{x}_n \to \mathbf{x}_0$ in l_∞ where $\mathbf{x}_0 = \{x_{0,1}, x_{0,2}, x_{0,3}, \ldots\}$, then for each $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} x_{n,k} = x_{0,k}$ but that the converse can fail.

8) Let $P_0 = [0,1]$. Let P_1 be obtained from P_0 by removing the open interval of length $\frac{1}{3}$ from the middle of P_0. Then construct P_2 from P_1 by removing open intervals of length $\frac{1}{3^2}$ from the two closed subintervals of P_1. In general, P_{n+1} is obtained from P_n by removing the open interval of length $\frac{1}{3^{n+1}}$ from the middle of each of the 2^n closed subintervals of P_n. Let

$$P = \bigcap_{n=0}^{\infty} P_n.$$

P is called the Cantor set.

a) A subset A of a metric space is **nowhere dense** if $int\bar{A} = \emptyset$. Show that P is closed and nowhere dense.

b) Show that P is uncountable. (Hint: You may use the fact that $x \in P$ if and only if we can express $x = \sum\limits_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n = 0, 2$.)