# PMATH 351 Assignment 3

Due: Friday November 4

**1 a)** Let $(X, d)$ be a metric space. Let $x_0 \in X$ be fixed. Define $F_{x_0} : X \to \mathbb{R}$ by

$$F_{x_0}(x) = d(x_0, x)$$

Show that $F_{x_0}$ is continuous.

**b)** Let $(X, \| \cdot \|)$ be a normed linear space. $F : X \to \mathbb{R}$ by

$$F(x) = \| x \|$$

Show that $F$ is continuous.

**2)** A function $f : (X, d_X) \to (Y, d_Y)$ is said to be uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$.

Let $f : (X, d_X) \to (Y, d_Y)$ be uniformly continuous. Show that if $\{x_n\}$ is Cauchy in $X$, then $\{f(x_n)\}$ is Cauchy in $Y$.

**3)** Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed linear spaces. Let $T : X \to Y$ be linear. We say that $T$ is bounded is

$$\sup_{\|x\|_X \leq 1} \{\| T(x) \|_Y\} < \infty.$$

In this case, we write

$$\| T \| = \sup_{\|x\|_X \leq 1} \{\| T(x) \|_Y\}.$$

Otherwise, we say that $T$ is unbounded.

**a)** Prove that the following are equivalent

i) T is continuous.

ii) T is continuous at 0.

iii) T is bounded.

**b)** Assume that $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear and that $L$ is represented by the matrix $A$. We let $\| A \| = \| L \|$.

i) Assume that

$$D = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix}$$

is a diagonal matix. Show that $\| D \| = \max_{i=1,\cdots,n} \{| d_i |\}$.

ii) Show that if

$$D = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix}$$

is a diagonal matix, then

$$\sup_{\|x\| \leq 1} \{| < Dx, x > |\} = \max_{i=1,\cdots,n} \{| d_i |\}.$$

iii) Let $U$ be an orthonormal $n \times n$ matrix. Show that if $x \in \mathbb{R}^n$, then $\| Ux \| = \| x \|$.

iv) Assume that $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear and that $L$ is represented by the matrix $A$. Show that $\| L \| = \| A \| = \sqrt{| \alpha |}$ where $\alpha$ is the largest eigenvalue of the matrix $A^t A$.

iv) Assume that $L : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

Find $\| A \|$. (You can use Maple or MATLAB if you like.)

**4)** Let $x_0 \in [0,1]$. Define $T_{x_0} : C[0,1] \to \mathbb{R}$ by

$$T_{x_0}(f) = f(x_0)$$

**a)** Show that $T_{x_0}$ is linear.

**b)** Show that as a map from $(C[0,1], \| \cdot \|_\infty) \to \mathbb{R}$, $T_{x_0}$ is bounded with $\| T_{x_0} \| = 1$.

**c)** Show that as a map from $(C[0,1], \| \cdot \|_1) \to \mathbb{R}$, $T_0$ is unbounded.

**5)** Define $T : C[0,1] \to \mathbb{R}$ by

$$T(f) = \int_0^1 x f(x) dx.$$

**a)** Show that $T$ is linear.

**b)** Show that if $\| f(x) \|_\infty \leq 1$, then $| T(f) | \leq \frac{1}{2}$.

**c)** Show that $T(1) = \frac{1}{2}$ and hence that $\| T \| = \frac{1}{2}$.

**6** Let $(X, d)$ and $(Y, d)$ be metric spaces. We say that a function $f : (X, d_X) \to (Y, d_Y)$ is bounded if $range(f) = \{f(x) \mid x \in X\}$ is bounded in $Y$. (Note: This is diffferent than the notion of boundedness introduced for linear maps between normed-linear spaces.)

Let

$$C_b(X, Y) = \{f : X \to Y \mid f \text{ is continuous and bounded}\}$$

Define a function $d_\infty$ on $C_b(X, Y) \times C_b(X, Y)$ by

$$d_\infty(f, g) = \sup_{x \in X} \{d_Y(f(x), g(x))\}$$

**a)** Show that $d_\infty$ determines a norm on $C_b(X, Y)$.

**b)** Show that if $(Y, d)$ is complete, then so is $(C_b(X, Y), d_\infty)$.

**7)** Let $f : \mathbb{R} \to \mathbb{R}$. Let

$$D(f) = \{x \in \mathbb{R} \mid f(x) \text{ is discontinuous at } x\}.$$

For each $n \in \mathbb{N}$, let $D_n = \{x \in \mathbb{R} \mid$ for every $\delta > 0$, there exists $y, z$ with $| x - y | < \delta, | x - z | < \delta$ but $| f(y) - f(z) | \geq \frac{1}{n}\}$.

**a)** Show that for each $n \in \mathbb{N}$, $D_n$ is closed.

**b)** A subset $A$ of a metric space is said to be an $F_\sigma$ set if $A = \bigcup_{n=1}^{\infty} F_n$ where each $F_n$ is closed. Show that $D(f)$ is an $F_\sigma$ set by showing that

$$D(f) = \bigcup_{n=1}^{\infty} D_n.$$