PMATH 351 Assignment 4

Due: Monday, December 5

1) ($*$) Let $(X,\| \cdot \|)$ be a normed linear space.

a) Prove that if $A \subset (X,\| \cdot \|)$ is compact and nonempty, then for each $x_0 \in X$ there exist a $y_0 \in A$ such that

$$\| x_0 - y_0 \| = \inf\{\| x_0 - y \| \mid y \in A\}.$$

b) Assume that X is finite dimensional. Prove that if $A \subset (X,\| \cdot \|)$ is closed and nonempty, then for each $x_0 \in X$ there exist a $y_0 \in A$ such that

$$\| x_0 - y_0 \| = \inf\{\| x_0 - y \| \mid y \in A\}.$$

c) A subset A of a vector space is said to be convex if $\alpha x + (1-\alpha)y \in A$ whenever $x, y \in A$ and $0 \leq \alpha \leq 1$.

Let $A \subseteq \mathbb{R}^2$ be convex and closed an let $x_0 \in A^c$. Show that if $\mathbb{R}^2$ is given the norm $\| \cdot \|_2$, then the point y_0 obtained in part b) above is unique but that this need not be the case if we use the norm $\| \cdot \|_\infty$.

d) Let $A \subseteq (X, d)$ be nonempty and let $x_0 \in X$. Define the distance from x_0 to A by $dist(x_0, A) = \inf\{d(x_0, a) \mid a \in A\}$. Show that the function $f(x) = dist(x, A)$ is continuous. (**Note: Do not hand in**.)

e) Given $A, B \subseteq X$ nonempty sets, define $dist(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$. Show that if A is closed, B is compact with $A \cap B = \emptyset$, then $dist(A, B) > 0$.

f) Show that even in $\mathbb{R}$, $e)$ can fail if you only assume that B is closed.

Let $P_n = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}$.

g) Let $f(x) \in C[0, 1]$. Show that there exists a polynomial $p(x) \in P_n$ such that

$$\| f(x) - p(x) \|_\infty \leq \| f(x) - q(x) \|_\infty$$

for any $q(x) \in P_n$.

h) Show that if $\{p_k(x)\}$ is a sequence of polynomials such that $\{p_k(x)\}$ converges uniformly to $f(x) = e^x$ on $[0, 1]$, then

$$\lim_{k\to\infty} degree(p_k(x)) = \infty.$$

2) $(*)$ Let (X, d_X) be a compact metric space. Let $f : (X, d_X) \to (Y, d_Y)$ be continuous, 1-1 and onto, prove that f^{-1} is also continuous.

3) Connectedness and Path Connectedness:

Let (X, d_X) be a metric space. A continuous path joining $x, y \in X$ is a continuous function $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. A subset U of X is path connected if for each $x, y \in A$, there exists a continuous path γ joining x, y with $\gamma(t) \in U$ for all $t \in [a, b]$.

a) Show that if $A \subset \mathbb{R}$, then A is path connected if and only if A is an interval. (One direction is the Intermediate Value Theorem.)

b) $(*)$ For each of the following subsets of $\mathbb{R}^2$ indicate whether or not the set is path connected. (You do not need to justify your answer)

i) $A_1 = \{(x, y) \mid x^2 + y^2 \le r\}$
ii) $A_2 = \{(x, y) \mid xy \ge 1 \text{ and } x > 1\} \bigcup \{(x, y) \mid xy \le 1 \text{ and } x \le 1\}$
iii) $A_3 = \{(x, y) \mid y = \sin(\frac{1}{x}),\ x \ne 0\} \bigcup \{(0, 0)\}$
iv) $A_4 = \{(x, y) \mid \text{either } x \in Q \text{ or } y \in Q\}$

c) Let $A \subseteq (X, d_X)$ be path connected. Let $f : A \to (Y, d_Y)$ be continuous. Show that $f(A)$ is path connetcted.

Let $A \subseteq (X, d_X)$. We say that A is *disconnected* if there exists two open sets U and V such that

i) $U \bigcap V \bigcap A = \emptyset$
ii) $U \bigcap A \ne \emptyset$ and $V \bigcap A \ne \emptyset$
iii) $A \subseteq U \bigcup V$

We say that A is *connected* if it is not disconnected.

d) Show that if $A \subseteq (X, d_X)$ is path connected, then it is connected. (Note: This shows that $\mathbb{R}^n$ is connected).

e) (*) Give an example of a set $A \subset \mathbb{R}^2$ that is connected but not path connected. (Hint: Look at b) above. You do not need to justify your choice.)

f) $(*)$ Let $A \subseteq (X, d_X)$ be connected. Let $f : A \to (Y, d_Y)$ be continuous. Show that $f(A)$ is connected.

4) a) $(*)$ Assume that $F \subset \mathbb{R}$ is closed and nowhere dense. Let

$$f(x) = \chi_F(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \in F^c \end{cases} .$$

Find $D(f)$.

b) $(*)$ Show that if $A \subset \mathbb{R}$ is F_σ and of first category, then there exists a function $f(x)$ on $\mathbb{R}$ with $D(f) = A$. (Hint: You may assume without proof that $A = \bigcup_{n=1}^{\infty} F_n$ where F_n is closed and nowhere dense.)

5) a) $(*)$ **Dini's Theorem:** Let (X, d) be a compact metric space. Let $\{f_n(x)\}$ be a sequence of continuous functions on X such that $f_n(x) \leq f_{n+1}(x)$ for each $n \in \mathbb{N}$ and $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $f(x)$ is continuous on X if and only if the sequence converges uniformly. (Hint: Let $\epsilon > 0$. Let $U_n = \{x \in X \mid f_n(x) > f(x) - \epsilon\}$ and show that $\{U_n\}$ is an open cover of X.)

b) $(*)$ Show that Dini's Theorem fails on $[0, \infty)$ by giving a sequence $\{f_n(x)\}$ of continuous functions on $[0, \infty)$ such that $f_n(x) \leq f_{n+1}(x)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = 1$ for each x but for which the convergence is not uniform.

6) Show that if $(X, \|\cdot\|)$ is an infinite dimensional Banach space, then X must have uncountable dimension.

7) Let $f(x)$ be continuous on $[0,1]$. Assume that

$$\int_0^1 f(x)\,dx = 0$$

and that

$$\int_0^1 f(x)x^n\,dx = 0$$

for each $n \in \mathbb{N}$. Show that $f(x) = 0$ for all $x \in [0,1]$.

8) a) Let $X = [0,1] \times [0,1] \subset (\mathbb{R}^2, \|\cdot\|_2)$. Let $f(x,y) \in C(X)$. For each $y \in [0,1]$ define $f_y(x) = f(x,y)$ for each $x \in [0,1]$. Show that $\mathcal{F} = \{f_y \mid y \in [0,1]\}$ is equicontinuous.

b) Show that the map $\Gamma : [0,1] \to (C[0,1], \|\cdot\|_\infty)$ given by

$$\Gamma(y) = f_y$$

is continuous.

b) Is $\mathcal{F}$ compact in $C(X)$? Explain your answer.

9) $(*)$ Let

$$\Psi = \{F(x,y) \in C([0,1] \times [0,1] \mid F(x,y) = \sum_{i=1}^{k} f_i(x)g_i(y)\}$$

where in the sum above the functions f_i and g_i are continuous on $[0,1]$. Show that Ψ is dense in $C([0,1] \times [0,1])$.

10) Let I be a closed ideal of of $C[0,1]$. (That is I is a closed subalgebra of $C[0,1]$ with the property that if $g(x) \in I$ and if $f(x) \in C[0,1]$, then $f(x)g(x) \in I$.)

a) Let $Z(I) = \{x \in [0,1] \mid f(x) = 0 \text{ for every } f \in I\}$. Show that $Z(I)$ is a closed subset of $[0,1]$.

b) Show that if $Z(I) = \emptyset$, then $I = C[0,1]$. (Hint: Show that there exists a function $f(x) \in I$ such that $f(x) > 0$ for every $x \in [0,1]$).

c) Let $A \subseteq [0,1]$ be closed. Let $I(A) = \{f \in C[0,1] \mid f(x) = 0 \text{ for every } x \in A\}$. Show that I is a maximal closed idea in $C[0,1]$ if and only if $I = I(\{x_0\})$ for some $x_0 \in [0,1]$.

(Recall: A closed ideal I is maximal if $I \neq C[0,1]$ and if J is any closed ideal containing I, then either $I = J$ or $J = C[0,1]$.)

11) Let $g(x)$ be continuous and strictly increasing on $[a,b]$. Let $f(x) \in C[a,b]$. Let $\epsilon > 0$. Then there exists constants $c_0, c_1, \ldots, c_n$ such that

$$\mid f(x) - \sum_{k=0}^{n} c_k g^k(x) \mid < \epsilon$$

for each $x \in [a,b]$.

12 a) (∗) **Fredholm Equation:** Assume that $K(x,y) \in C([a,b] \times [a,b])$ with $\| K(x,y) \|_\infty = M$. Show that if $\mid \lambda \mid M(b-a) < 1$ and if $\varphi(x) \in C[a,b]$, then the map $\Gamma : C[a,b] \to C[a,b]$ given by

$$\Gamma(f)(x) = \varphi(x) + \lambda \int_a^b K(x,y) f(y) dy$$

is contractive and hence that the integral equation

$$f(x) = \varphi(x) + \lambda \int_a^b K(x,y) f(y) dy$$

has a unique solution in $C[a,b]$.

b) Volterra Equation: Assume that $K(x,y) \in C([a,b] \times [a,b])$ with $\| K(x,y) \|_\infty = M$. Let $\lambda \in \mathbb{R}$ and $\varphi(x) \in C[a,b]$. Define $\Gamma : C[a,b] \to C[a,b]$ by

$$\Gamma(f)(x) = \varphi(x) + \lambda \int_a^x K(x,y) f(y) dy.$$

i) Show that for each $n \in \mathbb{N}$ that

$$\| \Gamma(f) - \Gamma(g) \|_\infty \leq \mid \lambda \mid^n M^n \frac{(b-a)^n}{n!}$$

and hence that $\Gamma^{(n)} = \Gamma \circ \Gamma \circ \cdots \circ \Gamma$ is contractive for large enough n.

ii) (*) Show that Γ has a unique fixed point and hence that the integral equation

$$f(x) = \varphi(x) + \lambda \int_a^x K(x,y) f(y) dy$$

has a unique solution in $C[a,b]$.

discontinuities).