PMATH 351 Assignment 3

Solutions

1 a) Let (X, d) be a metric space. Let $x_0 \in X$ be fixed. Define $F_{x_0} : X \to \mathbb{R}$ by

$$F_{x_0}(x) = d(x_0, x)$$

Show that F_{x_0} is continuous.

Solution

Observe that if $x, y, z \in X$, then the triangle inequality shows that $d(x, z) \leq d(x, y) + d(y, z) = d(x, y) + d(z, y)$ and hence that

$$d(x, z) - d(x, y) \leq d(z, y)$$

A similar calculation shows that

$$d(x, y) - d(x, z) \leq d(z, y)$$

so we have

$$\mid d(x, z) - d(x, y) \mid \leq d(z, y)$$

Let $x_0 \in X$ be fixed. Let $y \in X$ and let $\epsilon > 0$. If $d(z, y) < \epsilon$, then

$$\mid F_{x_0}(z) - F_{x_0}(y) \mid = \mid d(x_0, z) - d(x_0, y) \mid \leq d(z, y) < \epsilon$$

Hence F_{x_0} is (uniformly) continuous on X.

b) Let $(X, \| \cdot \|)$ be a normed linear space. $F : X \to \mathbb{R}$ by

$$F(x) = \| x \|$$

Show that F is continuous.

Solution

Since $d(0, x) = \| x \|$, it is easy to see that $F(x) = F_0(x)$ in the notation above and is therefore continuous.

2) A function $f : (X, d_X) \to (Y, d_Y)$ is said to be uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$.

Let $f : (X, d_X) \to (Y, d_Y)$ be uniformly continuous. Show that if $\{x_n\}$ is Cauchy in X, then $\{f(x_n)\}$ is Cauchy in Y.

Solution

Let $\{x_n\}$ be Cauchy in X and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $d_X(u, w) < \delta$, then $d_Y(f(u), f(w)) < \epsilon$. But as $\{x_n\}$ is Cauchy in X, we can find an N so that if $n, m \geq N$, then $d_X(x_n, x_m) < \delta$. It then follows that if $n, m \geq N$, then $d_Y(f(x_n), f(x_m)) < \epsilon$. This means that $\{f(x_n)\}$ is Cauchy in Y.

3) Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed linear spaces. Let $T : X \to Y$ be linear. We say that T is bounded is

$$\sup_{\|x\|_X \leq 1} \{\| T(x) \|_Y\} < \infty.$$

In this case, we write

$$\| T \| = \sup_{\|x\|_X \leq 1} \{\| T(x) \|_Y\}.$$

Otherwise, we say that T is unbounded.

a) Prove that the following are equivalent

i) T is continuous.

ii) T is continuous at 0.

iii) T is bounded.

Solution

$i) \Rightarrow ii)$ is immediate.

$ii) \Rightarrow iii)$

Assume that T is continuous at 0. Let $\epsilon = 1$. Then there exists a $\delta > 0$ such that if $\| x - 0 \| = \| x \| \leq \delta$ then $\| T(x) - T(0) \| = \| T(x) \| \leq \epsilon = 1$. (We can use $\leq \delta$ since once we find a δ_1 that works in the definition of continuity $\delta = \frac{\delta_1}{2}$ also works.)

Now assume that $\| x \| \leq 1$ and that $x \neq 0$. Then

$$w = \frac{\delta}{\| x \|} x$$

has norm δ, so that

$$\| T(w) \| \leq 1$$

But $\| T(w) \| = \| T(\frac{\delta}{\|x\|} x \|) = \frac{\delta}{\|x\|} \| T(x) \|$, so that

$$\frac{\delta}{\| x \|} \| T(x) \| \leq 1.$$

This shows that

$$\| T(x) \| \leq \frac{\| x \|}{\delta} \leq \frac{1}{\delta}$$

since $\| x \| \leq 1$. It is also clear that $0 = \| T(0) \| \leq \frac{1}{\delta}$. It follows that $\| T(x) \| \leq \frac{1}{\delta}$ whenever $\| x \| \leq 1$. We have shown that $\| T \| \leq \frac{1}{\delta}$ and hence that T is bounded.

iii) $\Rightarrow$ *i)*

Observe that if T is bounded, then for any $w \in X$ with $w \neq 0$ we have that $\frac{1}{\|w\|} w$ has norm 1. It follows that

$$\| T(\frac{1}{\| w \|} w) \| = \frac{1}{\| w \|} \| T(w) \| \leq \| T \|$$

From this we get that $\| T(w) \| \leq \| T \| \| w \|$. In fact, since $T(0) = 0$, we get that if T is bounded, then

$$\| T(w) \| \leq \| T \| \| w \| .$$

If $T(w) = 0$ for all w, then T is clearly continuous. Assume that $T \neq 0$. Then $\| T \| > 0$. Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{\|T\|}$. If $\| x - y \| < \delta$, then

$$\begin{aligned} \| T(x) - T(y) \| &= \| T(x-y) \| \\ &\leq \| T \| \| x - y \| \\ &< \| T \| (\frac{\epsilon}{\| T \|}) \\ &= \epsilon \end{aligned}$$

This shows that T is actually unifromly continuous on X.

b) Assume that $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear and that L is represented by the matrix A. We let $\| A \| = \| L \|$.

i) Assume that

$$D = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix}$$

is a diagonal matix. Show that $\| D \| = \max_{i=1,\cdots,n} \{| d_i |\}$.

Solution

Assume that $| d_k | = \max_{i=1,\cdots,n} \{| d_i |\}$. Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ with $\| x \| \leq 1$. Then

$$\begin{aligned} \| D(x) \| &= \| (d_1x_1, d_2x_2, \cdots, d_nx_n) \| \\ &= \sqrt{(d_1x_1)^2 + (d_2x_2)^2 + \cdots + (d_nx_n)^2} \\ &\leq \sqrt{(d_kx_1)^2 + (d_kx_2)^2 + \cdots + (d_kx_n)^2} \\ &= | d_k | \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\ &\leq | d_k | \end{aligned}$$

Hence $\| D \| \leq d_k$. However, If $x = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$ where the 1 is in the k-th spot, then $D(x) = (0, 0, \cdots, 0, d_k, 0, \cdots, 0)$ so $\| D(x) \| = | d_k |$. This shows that $\| D \| = | d_k |$.

ii) Show that if

$$D = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix}$$

is a diagonal matix, then

$$\sup_{\|x\| \leq 1} \{|< Dx, x >|\} = \max_{i=1,\cdots,n} \{| d_i |\}.$$

Solution

Assume that $| d_k |= \max_{i=1,\cdots,n} \{| d_i |\}$. Observe that if $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ with $\| x \|\leq 1$, then

$$\begin{aligned} |< Dx, x >| &= | d_1 x_1^2 + d_2 x_2^2 + \cdots + d_n x_n^2 | \\ &\leq | d_1 | x_1^2 + | d_2 | x_2^2 + \cdots + | d_n | x_n^2 \\ &\leq | d_k | (x_1^2 + x_2^2 + \cdots + x_n^2) \\ &\leq | d_k | \end{aligned}$$

However, if $x = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$ where the 1 is in the k-th spot, then $|< Dx, x >|=| d_k |$. It follows that

$$\sup_{\|x\|\leq 1} \{|< Dx, x >|\} = \max_{i=1,\cdots,n} \{| d_i |\}.$$

iii) Let U be an orthonormal $n \times n$ matrix. Show that if $x \in \mathbb{R}^n$, then $\| Ux \|=\| x \|$.

Solution

If U is orthonormal, then $U^t U = I_n$, where U^t is the transpose of U and I_n is the $n \times n$ identity matrix.

It follows from basic linear algebra that for any $x \in \mathbb{R}^n$

$$\| Ux \|^2 =< Ux, Ux >=< U^t Ux, x >=< x, x >=\| x \|^2$$

iv) Assume that $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear and that L is represented by the matrix A. Show that $\| L \|=\| A \|= \sqrt{| \alpha |}$ where α is the largest eigenvalue of the matrix $A^t A$.

Solution

Assume that $\| x \|\leq 1$. Then

$$\begin{aligned} \| Ax \|^2 &= < Ax, Ax > \\ &= < A^t Ax, x > \end{aligned}$$

It follows that

$$\| A \|= \sqrt{\sup_{\|x\|\leq 1} \{|< A^t Ax, x >|\}}$$

But A^tA is symmetric and hence is diagonalizable via an orthomormal matrix U. That is $U^tA^tAU = D$ where D is the diagonal matrix with the eigenvalues of A^TA on its diagonal. Since $\| x \|= 1$ if and only if $\| Ux \|= 1$, we get

$$\begin{aligned} \| A \| &= \sqrt{\sup_{\|x\|\leq 1} \{|< A^tAx, x >|\}} \\ &= \sqrt{\sup_{\|x\|\leq 1} \{|< A^tAUx, Ux >|\}} \\ &= \sqrt{\sup_{\|x\|\leq 1} \{|< U^tA^tAUx, x >|\}} \\ &= \sqrt{\sup_{\|x\|\leq 1} \{|< Dx, x >|\}} \\ &= \sqrt{| \alpha |} \end{aligned}$$

where α is the largest eigenvalue of A^tA.

v) Assume that $L : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

Find $\| A \|$. (You can use Maple or MATLAB if you like.)

Solution

First note that

$$A^tA = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}$$

is symmetric and hence diagonalizable. To find the eigenvalues we evaluate

$$det \begin{bmatrix} 5-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$$

to get the characteristic polynomial $\lambda^2 - 7\lambda + 9$. The roots of this polynomial are $\lambda = \frac{7\pm\sqrt{13}}{2}$. Since $\frac{7+\sqrt{13}}{2}$ is the largest eigenvalue, we get that

$$\| A \|= \sqrt{\frac{7+\sqrt{13}}{2}}.$$

4) Let $x_0 \in [0,1]$. Define $T_{x_0} : C[0,1] \to \mathbb{R}$ by

$$T_{x_0}(f) = f(x_0)$$

a) Show that T_{x_0} is linear.

Solution

For any scalars $\alpha, \beta \in \mathbb{R}$ and $f, g \in C[0,1]$, we have

$$\begin{aligned} T_{x_0}(\alpha f + \beta g) &= (\alpha f + \beta g)(x_0) \\ &= (\alpha f)(x_0) + (\beta g)(x_0) \\ &= \alpha f(x_0) + \beta g(x_0) \\ &= \alpha T_{x_0}(f) + \beta T_{x_0}(g) \end{aligned}$$

so T_{x_0} is linear.

b) Show that as a map from $(C[0,1], \| \cdot \|_\infty) \to \mathbb{R}$, T_{x_0} is bounded with $\| T_{x_0} \| = 1$.

Solution

Assume that $\| f \|_\infty \leq 1$. Then $| T_{x_0}(f) | = | f(x_0) | \leq \| f \|_\infty \leq 1$. Therefore, $\| T_{x_0} \| \leq 1$.

However, if $g(x) = 1$ for all $x \in [0,1]$, then $\| g \|_\infty = 1$ and $T_{x_0}(g) = g(x_0) = 1$, so $\| T_{x_0} \| = 1$.

c) Show that as a map from $(C[0,1], \| \cdot \|_1) \to \mathbb{R}$, T_0 is unbounded.

Solution

Let $f_n(x)$ be defined as follows. On $[0, \frac{1}{n}]$, $f(x) = 2n - 2n^2 x$. On $[\frac{1}{n}, 1]$, $f(x) = 0$. Then it is easy to see that $\| f_n \|_1 = 1$ but $T_0(f_n) = f_n(0) = 2n \to \infty$. Hence T_0 is unbounded.

5) Define $T : C[0,1] \to \mathbb{R}$ by

$$T(f) = \int_0^1 x f(x) dx.$$

a) Show that T is linear.

Solution

For any scalars $\alpha, \beta \in \mathbb{R}$ and $f, g \in C[0,1]$, we have

$$\begin{aligned}
T(\alpha f + \beta g) &= \int_0^1 (\alpha f + \beta g)(x)\,dx \\
&= \int_0^1 (\alpha f)(x) + (\beta g)(x)\,dx \\
&= \int_0^1 \alpha f(x)\,dx + \int_0^1 \beta g(x)\,dx \\
&= \alpha \int_0^1 f(x)\,dx + \beta \int_0^1 g(x)\,dx \\
&= \alpha T(f) + \beta T(g)
\end{aligned}$$

so T is linear.

b) Show that if $\| f(x) \|_\infty \leq 1$, then $| T(f) | \leq \frac{1}{2}$.

Solution

Assume that $\| f(x) \|_\infty \leq 1$,. Then

$$\begin{aligned}
| T(f) | &= | \int_0^1 x f(x)\,dx | \\
&\leq \int_0^1 | x f(x) | \; dx \\
&\leq \int_0^1 x \| f \|_\infty \; dx \\
&\leq \int_0^1 x\,dx \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, $\| T \| \leq \frac{1}{2}$.

c) Show that $T(1) = \frac{1}{2}$ and hence that $\| T \| = \frac{1}{2}$.

Solution

Let 1 denote the constant function with value 1 on $[0,1]$. Then $T(1) = \int_0^1 x \cdot 1\,dx = \int_0^1 x\,dx = \frac{1}{2}$. Since $\| 1 \|_\infty = 1$, we get that $\| T \| = \frac{1}{2}$.

6 Let (X, d) and (Y, d) be metric spaces. We say that a function $f : (X, d_X) \to (Y, d_Y)$ is bounded if $range(f) = \{f(x) \mid x \in X\}$ is bounded in Y. (Note: This is diffferent than the notion of boundedness introduced for linear maps between normed-linear spaces.)

Let

$$C_b(X, Y) = \{f : X \to Y \mid f \text{ is continuous and bounded}\}$$

Define a function d_∞ on $C_b(X, Y) \times C_b(X, Y)$ by

$$d_\infty(f, g) = \sup_{x \in X}\{d_Y(f(x), g(x))\}$$

a) Show that d_∞ determines a metric on $C_b(X, Y)$.

Solution

First observe that since $f(x)$ and $g(x)$ are both bounded on X, the set $f(X) \cup g(X)$ is also bounded in Y. This shows that $0 \leq d_\infty(f, g) = \sup_{x \in X}\{d_Y(f(x), g(x))\} < \infty$. Moreover, it is clear that the only way that $0 = d_\infty(f, g)$ is if $d_Y(f(x), g(x)) = 0$ for all $x \in X$. This clearly happens only if $f = g$. It is also clear from the symmetry of the definition that $d_\infty(f, g) = d_\infty(g, f)$.

To prove the triangle inequality, let $f, g, h \in C_b(X, Y)$ and let $x \in X$.

Then by the triangle inequality for d_Y, we get

$$d_Y(f(x), g(x)) \leq d_Y(f(x), h(x)) + d_Y(h(x), g(x)) \leq d_\infty(f, h) + d_\infty(h, g).$$

Since the choice of x above was arbitrary, this shows that

$$d_\infty(f, g) = \sup_{x \in X}\{d_Y(f(x), g(x))\} \leq d_\infty(f, h) + d_\infty(h, g).$$

Hence d_∞ satisfies the three properties of a metric.

b) Show that if (Y, d) is complete, then so is $(C_b(X, Y), d_\infty)$.

Solution

Let $\{f_n\}$ be a Cauchy sequence in $(C_b(X, Y), d_\infty)$. Then if $x_0 \in X$, and if $n, m \in \mathbb{N}$, we have

$$d_Y(f_n(x_0), f_m(x_0)) \leq d_\infty(f_n, f_m).$$

From this it is immediate that $\{f_n(x_0)\}$ is Cauchy in Y and is therefore convergent because Y is complete.

For each $x \in X$ define

$$f_0(x) = \lim_{n\to\infty} f_n(x)$$

We claim that $f_n \to f_0$ uniformly. To see this let $\epsilon > 0$. Choose N so that if $n, m \geq N$, then $d_\infty(f_n, f_m) < \frac{\epsilon}{2}$. Now let $n \geq N$ and pick $x \in X$.

Then $f_m(x) \to f_0(x)$ in Y. It follows by continuity (see Question 1 above) that

$$d_Y(f_n(x), f_0(x)) = \lim_{m\to\infty} d_Y(f_n(x), f_m(x))$$

But if $m \geq N$, we have $d_Y(f_n(x), f_m(x)) < \frac{\epsilon}{2}$. It follows that

$$d_Y(f_n(x), f_0(x)) < \frac{\epsilon}{2}$$

and hence that

$$d_\infty(f_n, f_m) \leq \frac{\epsilon}{2} < \epsilon.$$

We have just shown that $f_n \to f$ uniformly on X. Since each f_n is continuous, so is its uniform limit f_0. To see that f_0 is also bounded, we note that we can find an N such that for each $m > N$, $d_\infty(f_N, f_m) \leq 1$. Since $f_N(X)$ is bounded, there is an y_0 and a M such that $f_N(X) \subseteq B(y_0, M)$. Now for each $x \in X$, we have

$$\begin{aligned} d_Y(y_0, f_0(x)) &\leq d_Y(y_(0), f_N(x)) + d_Y(f_N(x), f_0(x)) \\ &= d_Y(y_(0), f_N(x)) + \lim_{m\to\infty} d_Y(f_N(x), f_m(x)) \\ &\leq M + 1 \end{aligned}$$

since $d_Y(f_N(x), f_m(x)) \leq 1$ for each $x \in X$.

The final observation is that uniform convergence is equivalent to convergence in the given metric. Hence $f_n \to f_0$ in d_∞ and $(C_b(X,Y), d_\infty)$ is complete.

7) Let $f : \mathbb{R} \to \mathbb{R}$. Let

$$D(f) = \{x \in \mathbb{R} \mid f(x) \text{ is discontinuous at } x\}.$$

For each $n \in \mathbb{N}$, let $D_n = \{x \in \mathbb{R} \mid$ for every $\delta > 0$, there exists y, z with $\mid x - y \mid < \delta, \mid x - z \mid < \delta$ but $\mid f(y) - f(z) \mid \geq \frac{1}{n}\}$.

a) Show that for each $n \in \mathbb{N}$, D_n is closed.

Solution

Let $\{x_k\}$ be a sequence in D_n with $x_k \to x_0$. Let $\delta > 0$. Then there exists a k_0 such that $x_{k_0} \in B(x_0, \delta)$. Let $d = \delta - \mid x_0 - x_{k_0} \mid > 0$. Since $x_{k_0} \in D_n$, there exists $y, z \in B(x_{k_0}, d)$ with $\mid f(y) - f(x) \mid \geq \frac{1}{n}$. However, $B(x_{k_0}, d) \subseteq B(x_0, \delta)$ so $y, z \in B(x_0, \delta)$ and hence $x_0 \in D_n$. This shows that D_n is closed.

b) A subset A of a metric space is said to be an F_σ set if $A = \bigcup\limits_{n=1}^{\infty} F_n$ where each F_n is closed. Show that $D(f)$ is an F_σ set by showing that

$$D(f) = \bigcup_{n=1}^{\infty} D_n.$$

Solution

First assume that $f(x)$ is discontinuous at x_0. Then there exists an ϵ_0 such that for each $\delta > 0$ there exists a y with $\mid x_0 - y \mid < \delta$ but $\mid f(x_0 - f(y) \mid \geq \epsilon_0$. If we choose n large enough so that $\frac{1}{n} < \epsilon_0$, then with x_0 acting as the z, we see that $x_0 \in D_n$. Hence $D(f) \subseteq \bigcup\limits_{n=1}^{\infty} D_n$

Now suppose that $x_0 \in D_n$. Let $\epsilon_0 = \frac{1}{2n}$. Given $\delta > 0$ we get $y, z \in B(x_0, \delta)$ with $\mid f(y) - f(z) \mid \geq \frac{1}{n}$. But then the triangle inequality shows that either $\mid f(y) - f(x_0) \mid \geq \frac{1}{2n} = \epsilon_0$ or $\mid f(x_0) - f(z) \mid \geq \frac{1}{2n} = \epsilon_0$. Hence for each $\delta > 0$ there exists a $w \in B(x_0, \delta)$ such that $\mid f(y) - f(x_0) \mid \geq \epsilon_0$. This shows that $x_0 \in D(f)$ and hence that $D(f) = \bigcup\limits_{n=1}^{\infty} D_n$. Since each D_n is closed, $D(f)$ is an F_σ set.