Theorem [Banach Contractive Mapping Theorem]

Let (X, d) be complete metrric space. Let $0 < k < 1$. Let $\Gamma : X \to X$ be such that $d(\Gamma(x), \Gamma(y)) \leq kd(x, y)$ for every $x, y \in X$. Then there exists a unique $x_0 \in X$ such that $\Gamma(x_0) = x_0$.

Proof

Let $x_1 \in X$. Then let $x_2 = \Gamma(x_1)$, $x_3 = \Gamma(x_2)$, and proceed recursively by defining

$$x_{n+1} = \Gamma(x_n).$$

Note that

$$d(x_3, x_2) = d(\Gamma(x_2), \Gamma(x_1) \leq kd(x_2, x_1).$$

Similarly,

$$d(x_4, x_3) = d(\Gamma(x_3), \Gamma(x_3) \leq kd(x_3, x_2) \leq k^2 d(x_2, x_1).$$

In fact, we can proceed inductively to show that

$$d(x_{n+1}, x_n) =\leq k^{n-1} d(x_1, x_2).$$

From this it follows that if $m < n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots d(x_{n+1}, x_n) \\ &\leq k^{m-2} d(x_2, x_1) + k^{m-1} d(x_2, x_1) + \cdots + k^{n-1} d(x_2, x_1) \\ &= k^{n-1} d(x_2, x_1)[k^{m-n-1} + k^{m-n-2} + \cdots + k + 1] \\ &= \frac{k^{n-1} d(x_2, x_1)}{1 - k} \end{aligned}$$

Since $k^n \to 0$, it follows that $\{x_n\}$ is Cauchy. As (X, d) is complete $\{x_n\}$ converges to some $x_0 \in X$.

Now, It is clear that Γ is continuous. As such we have that $\Gamma(x_n) \to \Gamma(x_0)$. But $\Gamma(x_n) = x_{n+1} \to x_0$, so it follows that

$$\Gamma(x_0) = x_0.$$

Finally assume that y_0 also satisfies $\Gamma(y_0) = y_0$. Then

$$d(x_0, y_0) = d(\Gamma(x_0), \Gamma(y_0)) \leq kd(x_0, y_0).$$

As $0 < k < 1$, this implies that $[d(x_0, y_0) = 0$ and hence that $x_0 = y_0$.

Theorem [BaireCategory Theorem]

Let (X, d) be complete metrric space. Let $\{U_n\}$ be a sequence of open dense sets. Then $\bigcap\limits_{n=1}^{\infty} U_n$ is dense in X.

Proof

Let W be open and non-empty. Then there exists an $x_1 \in X$ and $0 < r_1 < 1$ such that

$$B(x_1, r_1) \subseteq B[x_1, r_1] \subseteq W \cap U_1.$$

Next we can find $x_2 \in X$ and $0 < r_2 < \frac{1}{2}$ such that

$$B(x_2, r_2) \subseteq B[x_2, r_2] \subseteq B(x_1, r_1) \cap U_2.$$

We can then proceed recursively to find sequences $\{x_n\} \subseteq X$ and $\{r_n\} \subset \mathbb{R}$ with $0 < r_n < \frac{1}{n}$, and

$$B(x_{n+1}, r_{n+1}) \subseteq B[x_{n+1}, r_{n+1}] \subseteq B(x_n, r_n) \cap U_{n+1}.$$

Since $r_n \to 0$ and $B[x_{n+1}, r_{n+1}] \subseteq B[x_n, r_n]$, Cantor's Intersection Theorem implies that there exists an

$$x_0 \in \bigcap_{n=1}^{\infty} B[x_n, r_n.]$$

But then $x_0 \in B[x_1, r_1] \subseteq W$ and $x_0 \in B[x_n, r_n] \subseteq U_n$ for each $n \in \mathbb{N}$. This shows that

$$x_0 \in W \cap (\bigcap_{n=1}^{\infty} U_n).$$

Note: If asked to prove the Weierstrass Approximation Theorem you can give the proof as follows:

Theorem [Weierstrass Approximation Theorem.]
Let $f \in C[a,b]$. Then there exists a sequence $p_n(x)$ of polynomials such that $p_n(x) \to f(x)$ uniformly on $[a,b]$

Proof
First we note that withoot loss of generality we can assume that $[a,b] = [0,1]$ and that $f(0) = 0 = f(1)$. As such we may extend $f(x)$ to a uniformly continuous function on $\mathbb{R}$ by defining $f(x) = 0$ if $x \in (-\infty, 0] \cup [1, \infty)$.
Now let $Q_n(x) = c_n(1-x^2)^n$ where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)dx = 1.$$

Using the MeanValue Theoerem we can show that

$$(1-x^2)^n \geq 1 - nx^2$$

for all $x \in [0,1]$. As such

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^n dx &= 2\int_0^1 (1-x^2)^n dx \\ &\geq 2\int_0^{\frac{1}{\sqrt{n}}} 1 - nx^2 dx \\ &= \frac{4}{3\sqrt{n}} \\ &> \frac{1}{\sqrt{n}} \end{aligned}$$

and hence we have

$$c_n < \sqrt{n}.$$

Now if $0 < \delta < 1$, then for each $x \in [-1, \delta] \cup [\delta, 1]$ we have

$$c_n(1-x^2)^n \leq \sqrt{n}(1-\delta^2)^n.$$

Let

$$\begin{aligned} p_n(x) &= \int_{-1}^{1} f(x+t)Q_n(t)dt \\ &= \int_{-x}^{1-x} f(x+t)Q_n(t)dt \\ &= \int_{-1}^{1} f(u)Q_n(u-x)du \end{aligned}$$

From Leibnez's rule we have that

$$\frac{d^{2n+1}}{dx^{2n+1}}(p_n(x)) = \int_{-1}^{1} f(u)\frac{\partial^{2n+1}}{\partial x^{2n+1}}Q_n(u-x)du = 0.$$

It follows that p_n is a polynomial of degree $2n+1$ or less.

Let $\epsilon > 0$.Let $M = \| f \|_\infty$. Choose $0 < \delta < 1$ so that if $| x - y | < \delta$, then $| f(x) - f(y) | < \frac{\epsilon}{2}$. Now

$$\int_{-1}^{1} Q_n(t)dt = 1 \Rightarrow f(x) = \int_{-1}^{1} f(x)Q_n(t)dt = 1.$$

Moreover, if $x \in [0,1]$,

$$\begin{aligned} | p_n(x) - f(x) | &= | \int_{-1}^{1} [f(x+t) - f(x)]Q_n(t)dt | \\ &\leq \int_{-1}^{1} | f(x+t) - f(x) | Q_n(t)dt \\ &= \int_{-1}^{delta} | f(x+t) - f(x) | Q_n(t)dt + \int_{-\delta}^{\delta} | f(x+t) - f(x) | Q_n(t)dt + \int_{\delta}^{1} | f(x \\ &\leq 2M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} + 2M\sqrt{n}(1-\delta^2)^n \\ &= 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} \end{aligned}$$

Hence if we choose n large enough so that $4M\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{2}$, then

$$\| p_n - f \|_\infty < \epsilon.$$

In the proof of the next theorem you may assume 1) the WAT, 2) the Stone-Weierstrass Theorem: Lattice version and that if $\mathcal{A}$ is a subsalgebra of $C(x)$, then so is $\bar{\mathcal{A}}$.

Theorem [Stone-Weierstrass Theorem: Subalgebra version]

Assume that (X,d) is a compact metric space. Let $\mathcal{A}$ be a subalgebra of $C(X)$ for which

1) 1 in $\mathcal{A}$,

2) $\mathcal{A}$ is point separating.

Then $\bar{\mathcal{A}} = C(X)$.

Proof

Note that $\bar{\mathcal{A}}$ is also a subalgebra satisfying 1) and 2). Let $f \in \bar{\mathcal{A}}$. Since f is bounded, there exists an $M > 0$ such that $f(x) \in [-M, M]$ for all $x \in X$. Now let $\epsilon > 0$. Then we can find a polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ so that

$$\mid p(t) - \mid t \mid \mid < \epsilon$$

for all $t \in [-M, M]$. It follows that for each $x \in X$ we have

$$\mid p(f(x)) - \mid f(x) \mid \mid < \epsilon.$$

As such if $p \circ f = a_0 1 + a_1 f + \cdots a_n f^n$, then $p \circ f \in \bar{\mathcal{A}}$ and

$$\| p \circ f - \mid f \mid \|_\infty < \epsilon.$$

Since ϵ was arbitrary and$\bar{\mathcal{A}}$ is closed, this shows that $\mid f \mid \in \bar{\mathcal{A}}$. Since

$$f \wedge g = \frac{f + g + \mid f - g \mid}{2}$$

we have that $\bar{\mathcal{A}}$ satisfies the conditions of the SWT Lattice version, and hence that $\bar{\mathcal{A}} = C(X)$.