PMATH 352, Spring 2011

Solution of Homework #1

Problem 1. Let $V \subset \mathbb{C}$ be open and $z \in \mathbb{C}$.

(a) If $f : V \to \mathbb{C}$ is $\mathbb{C}$-differentiable at z, show that f is continuous at z.

(b) Prove the *product rule*: if $f, g : V \to \mathbb{C}$ are each $\mathbb{C}$-differentiable at z, then so too is fg with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

SOLUTION.

(a) We have for each $z \in V$

$$\lim_{w \mapsto z} \left| \frac{f(w) - f(z)}{w - z} - f'(z) \right| = 0.$$

By usual limit rules and the continuity of $w \mapsto |w|$ we have

$$\begin{aligned}
\lim_{w \mapsto z} |f(w) - f(z)| &= \left| \lim_{w \mapsto z} (f(w) - f(z)) + 0 \right| \\
&= \left| \lim_{w \mapsto z} (f(w) - f(z)) + \lim_{w \mapsto z} f'(z)(w - z) \right| \\
&= \lim_{w \mapsto z} \left(\left| \frac{f(w) - f(z)}{w - z} - f'(z) \right| |w - z| \right) \\
&= \lim_{w \mapsto z} \left| \frac{f(w) - f(z)}{w - z} - f'(z) \right| \lim_{w \mapsto z} |w - z| = 0.
\end{aligned}$$

This implies $\lim_{w - z} f(w) = f(z)$.

Altenatively, use 2. (a) to obtain the estimate

$$|f(w) - f(z)| \leq |a||w - z| + |E(w - z)|$$

where the right-hand side goes to 0 as $w \to z$.

(b) We have for $z \in V$ and $h \neq 0$ such that $z + h \in V$,

$$\begin{aligned}
&\frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\
&\qquad = \frac{f(z+h)g(z+h) - f(z)g(z+h) + f(z)g(z+h) - f(z)g(z)}{h} \\
&\qquad = \frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h}.
\end{aligned}$$

We take limit $h \to 0$ and note that $\lim_{h \to 0} g(z + h) = g(z)$ by (a), above, and use usual limit rules, to obatain the desired result. ∎

Problem 2. Let $V \subset \mathbb{C}$ be open and $z \in \mathbb{C}$.

(a) Show that $f : V \to \mathbb{C}$ is differentiable at z if and only if there is $a \in \mathbb{C}$ and a function $E : D(0, r) \to \mathbb{C}$ (where $r > 0$ is such that $D(z, r) \subset V$) such that

$$f(z + h) = f(z) + ah + E(h) \text{ for } h \in D(0, r), \quad E(0) = 0 \text{ and } \lim_{h \to 0} \frac{E(h)}{h} = 0.$$

(b) Prove that for the "error" function E, in (a), above, that for any $\epsilon > 0$ there is $\delta > 0$ such that for $|h| < \delta$ we have $|E(h)| \leq \epsilon |h|$.

(c) Prove the *chain rule*: If $f : V \to \mathbb{C}$ is differentiable at z, U is an open set containing $f(z)$ and $g : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $f(z)$, then the composition $g \circ f$ is $\mathbb{C}$-differentiable at z with

$$(g \circ f)'(z) = g'\big(f(z)\big)g'(z).$$

(d) Let $q : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be given by $q(z) = \frac{1}{z}$. Show that $q'(z)$ exists for each $z \neq 0$ and is equal to $-\frac{1}{z^2}$. Deduce from this, and from the differentiation rules above, the *quotient rule*: if $f, g : V \to \mathbb{C}$ are $\mathbb{C}$-differentiable at $z \in V$ and $g(z) \neq 0$, then f/g is $\mathbb{C}$-differentiable at z with

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

Solution.

(a) If f is $\mathbb{C}$-differentiable at z, set $a = f'(z)$ and define $E : D(0, r) \to \mathbb{C}$ by

$$E(h) = \begin{cases} \frac{f(z+h)-f(z)}{h} - f'(z) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}.$$

Then the desired result holds, by definition.

Conversely, if a, E are as in the question we obtain

$$\frac{f(z+h) - f(z)}{h} = a + \frac{E(h)}{h} \text{for } h \neq 0.$$

Since $\lim_{h\to 0} \frac{E(h)}{h} = 0$ we see that $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exist and equals a.

(b) Since $\lim_{h\to 0} \frac{E(h)}{h} = 0$, given $\epsilon > 0$ there is $\delta > 0$ (we may assume $\delta \leq r$) so that

$$\frac{|E(h)|}{|h|} < \epsilon \text{ whenever } 0 < |h| < \delta.$$

Hence, using additionally the fact that $E(0) = 0$ we find

$$|E(h)| \leq \epsilon|h| \text{ whenever } |h| < \delta.$$

(c) Let $E : D(0, r) \to \mathbb{C}$ be the error function for f at z as promised by (a); and $F : D(0, r') \to \mathbb{C}$ the error function for g at $f(z)$, as promised by (a). For suitably small $h \neq 0$ we have

$$\begin{aligned} g(f(z+h)) - g(f(z)) &= g\big(f(z) + f'(z)h + E(h)\big) - g(f(z)), \text{ by (a)} \\ &= g(f(z)) + g'(f(z))\big(f'(z)h + E(h)\big) + F\big(f'(z)h + E(h)\big) - g(f(z)), \text{ by (a)} \\ &= g'(f(z))\big(f'(z)h + E(h)\big) + F\big(f'(z)h + E(h)\big). \end{aligned}$$

Let $\epsilon > 0$ be given. By (b), find $\delta' \in (0, r')$ for which $|F(\eta)| \leq \frac{\epsilon}{2|f'(z)|+1}|\eta|$ for $|\eta| < \delta'$. Since $\lim_{h\to 0} E(h) = 0$ (see proof of 1. (a),above) we have that $\lim_{h\to 0}\big(f'(z)h + E(h)\big) = 0$. Hence there is $\delta \in (0, r)$ such that $|f'(z)h + E(h)| < \delta'$ for $|h| < \delta$. Thus for $0 < |h| < \delta$ we have

$$\left|\frac{g(f(z+h))-g(f(z))}{h}-g'(f(z))f'(z)\right| = \left|g'(f(z))\frac{E(h)+F\big(f'(z)h+E(h)\big)}{h}\right|$$
$$\le |g'(f(z))|\frac{|E(h)|}{|h|}+\frac{\big|F\big(f'(z)h+E(h)\big)\big|}{h}$$
$$\le |g'(f(z))|\frac{|E(h)|}{|h|}+\frac{\epsilon}{2|f'(z)|+1}\frac{\big|f'(z)h+E(h)\big|}{|h|}$$
$$\le \big(|g'(f(z))|+1\big)\frac{|E(h)|}{|h|}+\frac{\epsilon}{2}.$$

If we find $\delta_1 > 0$ so that $\frac{|E(h)|}{|h|} < \frac{\epsilon}{2\left(|g'(f(z))|+1\right)}$ whenever $0 < |h| < \delta_1$. Then if $0 < |h| < \delta_1$ we have

$$\left|\frac{g(f(z+h))-g(f(z))}{h}-g'(f(z))f'(z)\right| < \epsilon.$$

(d) We have

$$\frac{1}{z+h}-\frac{1}{z}=\frac{-h}{(z+h)z}.$$

Hence it follows that $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{z+h}-\frac{1}{z}\right)=-\frac{1}{z^2}$.

Using product rule and chain rule we obtain

$$\left(\frac{f}{g}\right)'=f'\frac{1}{g}+f\left(\frac{1}{g}\right)'=\frac{f'}{g}-f\frac{g'}{g^2}=\frac{f'g-fg'}{g^2}.$$

∎

Problem 3. Define $\phi(z)=i\dfrac{1-z}{1+z}$ for $z\neq -1$.

(a) Let $\lambda(z)=(1+iz)/(1-iz)$. Show that $\lambda(\phi(z))=z$, and that ϕ maps $\mathbb{C}\setminus\{-1\}$ one-to-one and onto $\mathbb{C}\setminus\{-i\}$.

(b) Show that ϕ maps the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ one-to-one and onto the upper half plane $\mathbb{H}=\{z:\operatorname{im} z>0\}$. (Hint: consider z in polar forms)

Solution.

(a) If $w=i(1-z)/(1+z)$, then $-iw(1+z)=1-z$; so $z=(1+iw)/1-iw)$. Thus, the inverse function of ϕ is λ. It follows that ϕ is one-to-one because it is determine by w. Also, every z is in the range of λ. Thus, ϕ maps $\mathbb{C}\setminus\{-1\}$ one-to-one and onto $\mathbb{C}\setminus\{-i\}$.

(b) Consider where a circle maps to. Let $z=r\cos\theta+r\sin\theta$. Compute

$$\phi(z) = i\frac{(1-r\cos\theta)-ir\sin\theta}{1+r\cos\theta)+ir\sin\theta}=i\frac{(1-r\cos\theta)-ir\sin\theta}{1+r\cos\theta)+ir\sin\theta}\cdot\frac{1+r\cos\theta)-ir\sin\theta}{1+r\cos\theta)-ir\sin\theta}$$
$$= i\frac{(1-r^2\cos^2\theta-r^2\sin^2\theta)+i(2r\cos\theta\sin\theta)}{(1+r\cos\theta)^2+r^2\sin^2\theta}=\frac{-r\sin 2\theta+i(1-r^2))}{(1+r\cos\theta)^2+r^2\sin^2\theta}.$$

Since the denominator is positive, this lies in the upper half plane if and only if $r<1$. By (a), the whole upper half plane is in the range of ϕ. So the unit disk must be mapped onto $\mathbb{H}$. ∎

Problem 4. Let $\mathbb{A} = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$, and let $f(z) = z + \frac{1}{z}$. Show that $f(\mathbb{A})$ is the interior of an ellipse. (Hint: consider the image of circles)

SOLUTION. Let $z = r\cos\theta + ir\sin\theta$. Then

$$f(z) = (r + 1/r)\cos\theta + i(r - 1/r)\sin\theta,$$

lies on the curve $x^2/(r+1/r)^2 + y^2/(r-1/r)^2 + 1$, which is an ellipse. To find the foci, compute

$$c^2 = (r + 1/r)^2 - (r - 1/r)^2 = 4.$$

So the foci are ± 2. Another way to describe the ellipse is

$$\{z||z - 2| + |z + 2| = 2(r + 1/r)\}.$$

When $r = 1$, we get $f(\cos\theta + \sin\theta) = 2\cos\theta$ which maps onto $[-2, 2]$.

Now we see that the images of the circles of radius $1 < r < 2$ are distinct ellipses with the same foci. They coincide with the image of the circles of radius $1/r$. The circle of radius 1 maps onto the line segment $[-2, 2]$. Since $r + 1/r$ is continuously monotone increasing on $r \geq 1$, and takes $[1, 2)$ onto $[4, 5)$, the image ellipses fill in the solid ellipse

$$f(\mathbb{A}) = \{z||z - 2| + |z + 2| < 5\}. \blacksquare$$

Problem 5.

Find all complex numbers z such that $z^8 + 16z^4 + 256 = 0$. Write your answers in the standard form.

SOLUTION. Let $t = z^4$. Then $t^2 + 16t + 256 = 0$. Using the quadratic formula, we get

$$t = \frac{-16 \pm \sqrt{256 - 4 \cdot 256}}{2} = -8 \pm 8\sqrt{3}i = 16e^{2\pi i/3}, 16e^{4\pi i/3}.$$

Thus,

$$z^4 = 16e^{2\pi i/3}, \text{ or } 16e^{4\pi i/3} \implies z = 2{\cdot}e^{\pi i/6}, 2{\cdot}e^{2\pi i/3}, 2{\cdot}e^{7\pi i/6}, 2{\cdot}e^{5\pi i/3}, 2{\cdot}e^{\pi i/3}, 2{\cdot}e^{5\pi i/6}, 2{\cdot}e^{4\pi i/3}, 2{\cdot}e^{11\pi i/6},$$

and therefore

$$z = \pm\sqrt{3} \pm i, \pm 1 \pm \sqrt{3}i. \blacksquare$$