PM 352 **Assignment 2 Solutions**

1. Suppose $f = u + iv$ on Ω where $u, v : \Omega \to \mathbb{R}$. Since f is real valued, $v(z) = 0$ for all $z \in \Omega$. Also, f is holomorphic so we know (by the Cauchy-Riemann equations)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Combining these two facts we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0.$$

It follows that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

and hence f is constant.

2. We calculate:

$$\begin{aligned} D_\zeta f(z) &= \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{f(z+t\zeta) - f(z)}{t} \\ &= \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{f(z+t\zeta) - f(z)}{t} \times \frac{\zeta}{\zeta} \\ &= \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{f(z+t\zeta) - f(z)}{t\zeta} \times \zeta \\ &= f'(z)\zeta, \end{aligned}$$

and we're done.

3. (a) Suppose that $R > 0$ and take any $0 \leq r < R$. We know that the series $(r^n|c_n|)_{n=0}^{\infty}$ is bounded. Hence there is a constant $M > 0$ such that $|c_n|r^n \leq M$ for all n. Hence

$$|c_n|^{1/n} \leq \frac{M^{1/n}}{r}.$$

Since $\lim_{n\to 0} M^{1/n} = 1$ it follows that

$$\limsup |c_n|^{1/n} \leq \frac{1}{r}.$$

As this is true for any $0 < r < R$ we have that $\limsup |c_n|^{1/n} \leq \frac{1}{R}$.
Now suppose that $\limsup |c_n|^{1/n} = t < \frac{1}{R}$. Choose an $\varepsilon > 0$ such that $t + \varepsilon < \frac{1}{R}$. By the definition of the *limit superior* we have that, for all but finitely many n,

$$|c_n|^{1/n} \leq t + \varepsilon.$$

Choose $t + \varepsilon < t_0 < \frac{1}{R}$. For all but finitely many n we have

$$|c_n|^{1/n}\frac{1}{t_0} \leq \frac{t+\varepsilon}{t_0} < 1.$$

Hence the sequence $(\frac{1}{t_0}^n|c_n|)$ is bounded. But $\frac{1}{t_0} > R$. This contradiction tells us that $\limsup |c_n|^{1/n} = R$.
Now suppose that $R = \infty$. Using the same argument as above we have that $\limsup |c_n|^{1/n} \leq \frac{1}{r}$ for any $0 < r < \infty$. Hence $\limsup |c_n|^{1/n} = 0$.
Finally, we will show that $R = 0$ if and only if $(|c_n|^{1/n})_{n=0}^{\infty}$ is unbounded. Suppose $(|c_n|^{1/n})_{n=0}^{\infty}$ is bounded. Then we can find $L > 0$ such that for all but finitely many n we have

$$L|c_n|^{1/n} < 1.$$

Hence $(L^n|c_n|^{1/n})_{n=0}^{\infty}$ is bounded and $R \geq L \neq 0$. The converse follows from the $R > 0$ case (we showed $R \neq 0$ implies that $(|c_n|^{1/n})_{n=0}^{\infty}$ is bounded).

(b) Choose $0 < r < 1/L$ if $L \neq 0$ and $r > 0$ in the case when $L = 0$. In either case we have $rL < 1$. Choose $\varepsilon > 0$ so that $rL + \varepsilon < 1$. By hypothesis there is an $N \in \mathbb{N}$ so that, for $n \geq N$, we have

$$\left| \frac{r|c_{n+1}|}{|c_n|} - rL \right| < \varepsilon.$$

Hence we have

$$-\varepsilon + rL < \frac{r|c_{n+1}|}{|c_n|} < \varepsilon + rL < 1.$$

He have now that, for n large enough, $r^{n+1}|c_{n+1}| < r^n|c_n|$. Since, after a finite number terms, the sequence $(r^n|c_n|)_{n=0}^{\infty}$ is decreasing we must have that $(r^n|c_n|)_{n=0}^{\infty}$ is bounded. Letting R be as in 3.(a), it follows $R = 1/L$ when $L \neq 0$ and $R = \infty$ when $L = 0$.

4. (a) For each $n \in \mathbb{N}$ define b_n as

$$b_n = \sum_{k=0}^{n} c_k d_{n-k}.$$

and let fg be the power series $(fg)(z) = \sum b_n z^n$. We will show that $(fg)(z)$ has radius of convergence at least $\min(R, S)$ and that the polynomials $f_N g_N$ converge to fg (where f_N is the sum of the first N terms of the power series f, and similarly for g).

First we make an important observation: If $(a_n)_{n=0}^{\infty}$ is a sequence of complex numbers and $T = \sup\{r \geq 0 : \ (r^n c_n)_{n=0}^{\infty}$ is bounded in $\mathbb{C}\}$, then T is the radius of convergence for the power series $\sum a_n z^n$. To see this, take $|z| < r < T$. Then $(|z|^n|a_n|)_{n=0}^{\infty}$ is bounded, say $|z|^n|a_n| \leq M$. We have now that

$$|z|^n|a_n| \leq |a_n||r|^n \leq |a_n| \left(\frac{r}{R}\right)^n R^n \leq M \left(\frac{r}{R}\right)^n.$$

It follows that $f(z)$ is absolutely convergent for $|z| < T$ (by comparing with a geometric series). That $\sum a_n z^n$ is not convergent when $|z| > T$ is immediate since $(|z|^n a_n)_{n=0}^{\infty}$ is not bounded. Hence question 3 has given us two useful ways to calculate the radius of convergence of a power series. We will use 3. (a) in what follows.

Now choose any $0 < s < \min(R, S)$. Since $(s^n c_n)_{n=0}^{\infty}$ and $(s^n d_n)_{n=0}^{\infty}$ are bounded we can find a constant C so that

$$|c_n| \leq \frac{C}{s^n} \text{ and } |d_n| \leq \frac{C}{s^n}.$$

It follows that

$$|b_n| \leq \frac{(n+1)C^2}{s^n}$$

and hence

$$|b_n|^{1/n} \leq \frac{(n+1)^{1/n}C^{2/n}}{s}.$$

Since $(n+1)^{1/n}C^{2/n} \to 1$ it follows that

$$\limsup |b_n|^{1/n} \leq \frac{1}{s}.$$

As this works for any $0 < s < \min(R, S)$ it follows from question 3 (a) and the preceding paragraph that the radius of convergence of $(fg)(z)$ is at least $\min(R, S)$.

Note that we have also shown that for $|z| < \min(R, S)$ the series $\sum_{n=N+1}^{\infty} \sum_{k=0}^{N} |c_k||b_{n-k}||z^n|$ is finite for any N. We have

$$|(fg)_N(z) - f_n(z)g_N(z)| \leq \sum_{n=N+1}^{\infty} \sum_{k=0}^{N} |c_k||b_{n-k}||z^n|$$

and since the right hand side can be made arbitrarily small (by increasing N) we have

$$f(z)g(z) = \lim f_N(z)g_N(z) = (fg)(z).$$

(b) The power series expansion of exp is $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$. That $\exp(w + z) = \exp(w)\exp(z)$ follows immediately from 4.(a), using the binomial theorem to expand $(w + z)^n$.
We show the first identity, the second is similar. Expanding the righthand side we get

$$\begin{aligned}
\cos(w)\cos(z) - \sin(w)\sin(z) &= \left(\frac{e^{iw} + e^{-iw}}{2}\right)\left(\frac{e^{iz} + e^{-iz}}{2}\right) - \left(\frac{e^{iw} - e^{-iw}}{2i}\right)\left(\frac{e^{iz} - e^{-iz}}{2i}\right) \\
&= \frac{1}{4}[e^{iw}e^{iz} + e^{iw}e^{-iz} + e^{-iw}e^{iz} + e^{-iw}e^{-iz} \\
&+ e^{iw}e^{iz} - e^{iw}e^{-iz} - e^{-iw}e^{iz} + e^{-iw}e^{-iz}] \\
&= \frac{e^{iw}e^{iz} + e^{-iw}e^{-iz}}{2} \\
&= \frac{e^{i(w+z)} + e^{-i(w+z)}}{2} = \cos(w + z).
\end{aligned}$$

5. (a) Omitted
(b) Let $w = x + i0$ where $x < 0$ be a point on the negative x-axis and let $w_n = |x|e^{i(1/n-\pi)}$. Since $e^{i\pi} = -1$ we have that $w_n \to w$ as $n \to \infty$.
Let $z_n = |x|e^{i(1/n=\pi)}$, then $z_n \to w$ by the same reasoning. However $\arg(w_n) = \pi - 1/n \to \pi$ and $\arg(z_n) = 1/n - \pi \to -\pi$. It follows that arg is not continuous on the negative x-axis.
(c) Take $w = x + iy \in K$, so $y > 0$. Alternatively $w = |w|e^{i\theta}$ where $0 < \theta < \pi$. It follows that

$$\cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}},$$

hence we define the logarithm as

$$\log(w) = \frac{1}{2}\log(x^2 + y^2) + i \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

for $w = x + iy \in K$.
To show that log is holomorphic on K you just need to check the Cauchy-Riemann equations. Recalling that

$$\frac{d}{dt}(\arccos(t)) = \frac{-1}{\sqrt{1 - t^2}}$$

for $-1 < t < 1$, you should get

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial y} = \frac{-y}{x^2 + y^2} \qquad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}.$$

(d) By the same reasoning as above we define the logarithm as

$$\log(w) = \frac{1}{2}\log(x^2+y^2) - i\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

for $w = x + iy \in L$. Again, we check that this satisfies the Cauchy-Riemann equations to show that it is holomorphic on L.

(e) It follows by how we defined log on the cut plane that $e^{\log(z)} = z$ for $z = x + iy$ and $x \nleq 0$. Differentiating we get

$$1 = e^{\log(z)}(\log(z))' = z(\log(z))'.$$

Hence $(\log(z))' = 1/z$.