PMath 352
Asg. 3 Solutions

Problem 1

a) Take any $r>0$. We can choose $n_0 \in \mathbb{N}$ s.t. $r(n_0+1)>2$

Letting $a_n = n!$, consider $(r^n a_n)_{n\geq 0}$

Since $r^{n+1}a_{n+1} = r(n+1)r^n a_n$ we have, for $n \geq n_0$

$$r^{n+1}a_{n+1} > 2r^n a_n$$

$\Rightarrow (r^n a_n)_{n\geq 0}$ is unbounded

$\therefore$ Radius of conv. for $\sum n! z^n$ is 0

(see Asg. 2 Q3)

b) $\frac{(n+1)^3}{n^3} = \left(1+\frac{1}{n}\right)^3 \Rightarrow \lim\limits_{n\to\infty} \frac{(n+1)^3}{n^3} = 1$

$\therefore$ Rad of conv. for $\sum n^3 z^n$ is 1

(See Asg. 2 Q3)

c) $\lim\limits_{n\to\infty} \left(\left(\frac{n-2}{n}\right)^n\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{n-2}{n} = \lim\limits_{n\to\infty} 1-\frac{2}{n} = 1$

$\therefore$ Rad of conv for $\sum \left(\frac{n-2}{n}\right)^n z^n$ is 1

d) $$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+\frac{1}{n}}\right)^n$$

Recall: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\therefore \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e}$

$\therefore$ Rad of conv for $\sum \frac{n!}{n^n} z^n$ is e.

Problem 3

$$\frac{1}{(2z-1)(z+2)} = \left(\frac{1}{(2z-1)}\right)\left(\frac{1}{z+2}\right) =: f(z)$$

$$\frac{1}{2z-1} = \frac{-1}{1-2z} = -\sum_{n=0}^{\infty} 2^n z^n \qquad (|z| < \tfrac{1}{2})$$

$$\frac{1}{z+2} = \frac{1}{2}\,\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\,\frac{1}{1-(-\frac{z}{2})} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n$$

$$(|z| < 2)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n$$

By Asg. 2 Q4

$$f(z) = \sum_{n=0}^{\infty} b_n z^n$$

Where $b_n = \sum_{k=0}^{n} (-2^k) \cdot \frac{(-1)^{n-k}}{2^{n-k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^{n-k+1}}{2^{n-2k+1}}$

with radius of conv. at least $\frac{1}{2}$.

Problem 4 (a)

$$\psi(z) = \frac{\alpha z+\beta}{\gamma z+\delta}, \quad \alpha\delta-\beta\gamma>0, \ \alpha,\beta,\gamma,\delta \in \mathbb{R}$$

To show that ψ maps $\mathbb{H}$ into $\mathbb{H}$ observe that

$$\operatorname{Im}\psi(z) = \frac{(ad-bc)y}{|cz+d|^2} \qquad (z=x+iy).$$

Now suppose $\psi(z_1)=\psi(z_2)$ for some $z_1, z_2 \in \mathbb{H}$

$$\Rightarrow 0 = \psi(z_1)-\psi(z_2)$$

$$\Rightarrow 0 = (\alpha z_1+\beta)(\gamma z_2+\delta) - (\alpha z_2+\beta)(\gamma z_1+\delta)$$

$$= \alpha\gamma z_1 z_2 + \alpha\delta z_1 + \beta\gamma z_2 + \beta\delta - (\alpha\gamma z_1 z_2 + \alpha\delta z_2 + \beta\gamma z_1 + \beta\delta)$$

$$= (z_1-z_2)(\alpha\delta-\beta\gamma)$$

Since $\alpha\delta-\beta\gamma>0$, we must have $z_1=z_2$

$\therefore$ ψ is one-to-one.

Now let $\psi' = \dfrac{\delta z-\beta}{-\gamma z+\alpha}$.

Show that $\psi'\circ\psi(z) = z$. ~~and t~~

The same argument as before shows ψ' is one-to-one.

Hence ψ is onto.

b) Suppose ψ is a bijective map $\mathbb{H} \to \mathbb{H}$

$\psi = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma \neq 0$.

ψ necessarily leaves the real line invariant, and so $\psi(\bar{z}) = \overline{\psi(z)}$ for all $z \in \mathbb{C}$

If $\bar{\psi}$ is the map $\bar{\psi}(z) = \overline{\psi(\bar{z})}$

then we have shown $\psi(z) = \bar{\psi}(z)$, thus, up to a scalar λ, ψ and $\bar{\psi}$ have the same matrix

(where the matrix of ψ is $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$).

So $\bar{A} = \lambda A$ ($\bar{A}$ the matrix for $\bar{\psi}$)

By above, λ satisfies $\lambda\bar{\lambda} = 1$, so

$$\lambda = e^{i\theta}$$

Replacing A with $e^{\frac{i\theta}{2}} A$ will give the same Möbius transform ψ, with real parts.

Problem 5

a) ~~Since complex con~~

Suppose $T(z) = e^{i\theta} \dfrac{z-a}{1-\bar{a}z}$.

Since complex conjugates have the same modulus we have

$$|T(e^{is})| = |e^{i\theta}| \left| \frac{e^{is}-a}{1-\bar{a}e^{is}} \right| = \left| \frac{e^{is}-a}{e^{-is}-\bar{a}} \right| = 1$$

$\Rightarrow$ T takes the unit circle to itself.

$|T(0)| = |a| < 1$

$\Rightarrow$ T takes $\mathbb{D}$ to itself.

Conversely suppose S is a Möbius map taking $\mathbb{D}$ to $\mathbb{D}$. So $S(0) = a$, some $a \in \mathbb{D}$.

Let T be as above so that

$$TS(0) = 0$$

$TS(z) = \dfrac{az+b}{cz+d}$, with $b = 0$. WLOG take $a = 1$.

$1 = |TS(e^{it})| = \dfrac{1}{|ce^{it}+d|}$ $\Rightarrow |ce^{it}+d| = 1 \quad \forall\, t \in \mathbb{R}$

$\Rightarrow$ Either $d=0$ and $|c|=1$ or $c=0$ and $|d|=1$.

If $d=0$ then TS is constant. (contradiction!)

$\Rightarrow c=0$. $\therefore TSz = \bar{d}z$. $\Rightarrow S^{-1} = dT$ is of desired form.

b) Translate C_1 so that it is on the unit circle.

If this translates C_2 to be outside the unit circle, then apply

$$z \to \frac{1}{z}$$

so that it lies inside $\mathbb{D}$.

Now rotate so that the centre of C_2 is on $(-1,1)$. Let a be the centre.

If $a=0$, we are done.

Otherwise, denoting by C_2' the image of C_2 under all these transformations, let r be the radius of C_2'.

Apply the map $T(z) = \frac{z-b}{1-bz}$ for $b \in (-1,1)$

T preserves the unit circle and $\mathbb{R}$. Thus, as C_2' intersects $\mathbb{R}$ orthogonally, so does TC_2'

$\Rightarrow TC_2'$ has centre on $(-1,1)$

The intersection points are $T(a \pm r)$

Solve $T(a+r) = -T(a-r)$ to find suitable b.