PM 352 **Assignment 4 Solutions**

1. Let $f(z) = 2/(z^2-1)$, then

$$f(z) = \frac{2}{z^2-1} = \frac{1}{z-1} - \frac{1}{z+1}.$$

Looking at each summand separately we have:

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n, \text{ when } |z| < 1,$$

and

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n, \text{ when } |z| < 1.$$

Thus the power series expansion of f is

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(-1)^{n+1} - \sum_{n=0}^{\infty} z^n \\ &= \sum_{n=0}^{\infty}[(-1)^{n+1} - 1]z^n = -2\sum_{n=0}^{\infty} z^{2n}, \end{aligned}$$

with radius of convergence 1.

2. (a) Let $f(z) = \cos(z)$. Note that $f'''(z) = \sin(z)$, thus by Cauchy's Integral Formula

$$\frac{3!}{2\pi i}\int_\gamma \frac{\cos(z)}{z^4}dz = f'''(0) = \sin(0) = 0.$$

Hence $\int_\gamma \frac{\cos(z)}{z^4}dz = 0$.

(b) Let $f(z) = e^{z^2}$. By Cauchy's Integral Formula

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{z^2}}{z^2}dz = f'(0) = 2(0)e^{0^2} = 0.$$

Hence $\int_\gamma \frac{e^{z^2}}{z^2}dz = 0$.

(c) Let $f(z) = \frac{z^2-1}{z^2+1}$. The function f has poles of order 1 at i and $-i$ (both are in the circle $|z| = 2$). Calculating the residue at the poles we have

$$\text{Res}(f, i) = i \text{ and } \text{Res}(f, -i) = -i.$$

The Residue Theorem tells us

$$\int_\gamma f(z)dz = 2\pi i(i + (-i)) = 0.$$

3. This follows immediately by the Maximum Modulus Principle. Here is another proof using Cauchy's Integral Formula. For any a in the disk,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = 0$$

where γ is the path around the boundary of the disk. It follows that $f(z) = 0$ on the disk.

4. First to show that u has the mean-value property. Let γ be the circle around p of radius $r < R$. Using Cuachy's Integral formula:

$$\begin{aligned} f(p) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-p} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(p+re^{it}) ire^{it}}{p + re^{it} - p} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(p+re^{it}) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(p+re^{it}) dt + i \frac{1}{2\pi} \int_0^{2\pi} v(p+re^{it}) dt, \end{aligned}$$

hence $u(p) = (1/2\pi) \int_0^{2\pi} u(p+re^{it}) dt$.

Now let M be the maximum value u takes on γ and let m be the minimum. Then

$$u(p) = \frac{1}{2\pi} \int_0^{2\pi} u(p+re^{it}) dt \leq \frac{1}{2\pi} \int_0^{2\pi} M dt = M.$$

A similar inequality holds for m, and we get $m \leq u(p) \leq M$.

If u is non-constant on the boundary then there is some $t \in [0, 2\pi]$ such that $u(p + re^{it}) < M$. Hence we have (using the hint)

$$u(p) = \frac{1}{2\pi} \int_0^{2\pi} u(p+re^{it}) dt < \frac{1}{2\pi} \int_0^{2\pi} M dt = M.$$

Now since M is attained at some $s \in [0, 2\pi]$ (by the extreme value theorem), we have that $u(p) < u(p + re^{is})$. A similar argument applied to m gives the other inequality.

5. We have$|f(z)| \leq M(1 + |z|^n)$. Take $k > n$, using Cauchy's Estimate for $f^{(k)}$ in a disk of radius R about the origin we get

$$|f^{(k)}(0)| \leq k! \frac{(M + MR^n)}{R^k}.$$

Taking the limit as $R \to \infty$ we get

$$|f^{(k)}(0)| \leq \lim_{R \to \infty} k! \frac{(M + MR^n)}{R^k} = 0.$$

Hence the power series of f about 0 is a polynomial of degree at most n. Since f is entire it follows that f is a polynomial of degree at most n.