PM 352 **Assignment 5 Solutions**

1. We could try to do this using the maximum modulus principle, but there is a much easier way. For any $z \in \mathbb{C}$, $|z+2|$ measures the distance from z to the point -2. By inspection the closest point in the triangle to -2 is $z = 0$ and the furthest is $z = 2$. Thus $|z+2|^2$ is minimized at $z = 0$ as $|0+2|^2 = 4$; and $|z+2|^2$ is maximized at $z = 2$ as $|2+2|^2 = 16$.
2. Take any closed curve α in Ω. We want to show that α is a homotopic to a point. WLOG we will view α as a function $\alpha : [0,1] \to \Omega$. Since Ω is star-shaped there is a point $p \in \Omega$ such that for any $t \in [0,1]$ the straight line from $\alpha(t)$ and p is in Ω. The points on this line can be described as $(1-s)\alpha(t) + sp$ for $s \in [0,1]$. For each $s \in [0,1]$ we define the curve α_s as

$$\alpha_s(t) = (1-s)\alpha(t) + sp$$

for $t \in [0,1]$. The map $\alpha_s : [0,1] \to \Omega$ is continuous since it is a linear combination of continuous functions, and is closed since $\alpha_s(0) = \alpha_s(1)$. Since $\alpha_1 = \alpha$ and $\alpha_0 = p$, α is homotopic to the point p.
3. (a) The closed curve γ is homotopic to a point $p \in \Omega$, thus $\text{Ind}_\gamma \omega = \text{Ind}_\alpha \omega$, where α is the closed curve that is constantly p. To see that $\text{Ind}_\alpha \omega = 0$ just look at the definition of the winding number (any integral over a path of length 0 will be 0).

   (b) Choose $z_0 \in \Omega$ define $h(z) = \int_\gamma f(w)dw$ where γ is a piecewise smooth curve from z_0 to z. Suppose δ is another piecewise smooth path from z_0 to z. Let δ^{-1} be the path from z to z_0, determined by δ (i.e. $\delta^{-1}(t) = \delta(1-t)$). Then $\alpha := \gamma + \delta^{-1}$ is a closed path in Ω. By Cauchy's Theorem we have

$$0 = \oint_\alpha f(w)dw = \int_\gamma f(w)dw - \int_\delta f(w)dw,$$

   thus we have $\int_\gamma f(w)dw = \int_\delta f(w)dw$. Hence, $h(z)$ is independent of the choice of γ. By what you've seen in class $h'(z) = f(z)$ and so h is the desired primitive.
4. Since Ω is simply connected, f analytic and non-vanishing, there is a branch of $\log(f(z))$ on Ω, i.e. there is a holomorphic function h on Ω such that $e^{h(z)} = f(z)$ for all $z \in \Omega$. Let $g(z) = e^{h(z)/n}$. Then $g(z)^n = e^{h(z)} = f(z)$.
5. First we will see how many zeros of $p(z)$ are in the open unit disk. Let $f(z) = -8z^2$ and $h(z) = z^8 + 2z^4 + 1$, so that $p(z) = f(z) + h(z)$. Now, for any $|z| = 1$

$$|h(z)| \leq |z|^8 + 2|z|^4 + 1 = 4 < 8 = |f(z)|.$$

Now by Rouché's Theorem, $p(z)$ and $f(z)$ have the same number of zeros in the open unit disk. Thus $p(z)$ has 2 zeros in the unit cirlce.

If we now calculate how many zeros p has in the disk $|z| < 2$ and subtract 2, then we will have found how many zeros are in $1 \leq |z| < 2$. However, the unit circle is not in $\mathbb{A}$ so we don't want to count the zeros there. Let's do the above calculation again with $|z|$ slightly larger than 1, say $|z| = 1.1$. Then

$$|h(z)| \leq |z|^8 + 2|z|^4 + 1 < 6.1 < 9.5 < |f(z)|.$$

Thus $p(z)$ has 2 zeros in the open disk $|z| < 1.1$. It follows that p has no zeros on the unit circle.

We now want to find how many zeros p has in the disk $|z| < 2$. Let $f(z) = z^8$ and $h(z) = 2z^4 - 8z^2 + 1$. Then when $|z| = 2$

$$|h(z)| \leq 2|z|^4 + 8|z|^2 + 1 = 65 < 256 = |f(z)|.$$

Again by Rouché's Theorem, we get that $p(z)$ has 8 zeros in the open disk $|z| < 2$. Thus p has $8 - 2 = 6$ zeros in $\mathbb{A}$.

6. Take any $R > 0$. We have by Cauchy's integral formula

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z^{n+1}} dz$$

where γ_R is the circle of radius R centred at 0. Since we $\lim_{|z|\to\infty} f(z)/z = 0$ it follows that $\lim_{|z|\to\infty} f(z)/z^n = 0$ for all positive n. Hence we have

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{f(z)}{z^{n+1}} dz = 0.$$

It follows that $f^{(n)}(0) = 0$ for all $n > 0$. Hence f has a constant power-series expansion around 0. Since f is entire, it follows that f is constant.

7. (a) By the product rule

$$p'(z) = c \sum_{i=1}^{k} n_i (z - a_i)^{n_i - 1} \prod_{j \neq i} (z - a_j)^{n_j}.$$

Thus we get

$$\frac{p'(z)}{p(z)} = \sum_{i=1}^{k} \frac{n_i}{z - a_i}.$$

(b) Suppose $p'(b) = 0$. If b is also a zero for p then $b = a_i$ for some i, and thus this it is a convex combination of $a_1, \ldots, a_k$.

Suppose that $p(b) \neq 0$. By our formula we have

$$0 = \sum_{i=1}^{k} \frac{n_i}{b - a_i} = \sum_{i=1}^{k} \frac{n_i \overline{(b - a_i)}}{|b - a_i|^2}.$$

We can write this as

$$\sum_{i=1}^{k} \frac{n_i (b - a_i)}{|b - a_i|^2} = 0,$$

since $\overline{0} = 0$. We have now that

$$\sum_{i=1}^{k} \frac{n_i b}{|b - a_i|^2} = \sum_{i=1}^{k} \frac{n_i a_i}{|b - a_i|^2}.$$

Let

$$t_i = \frac{n_i |b - a_i|^{-2}}{\sum_{j=1}^{k} n_j |b - a_j|^{-2}}.$$

Then $\sum t_i = 1$ and $b = \sum t_i a_i$.