1. (a) Let $f(z) = \frac{z^2}{(z^2+1)^2}$. For each $R > 0$ let $\gamma_R = [-R, R] \cup \{Re^{it} : 0 \le t \le \pi\}$. For R large enough, the integral of f around the semicircle of radius R is bounded above by

$$\pi R \frac{R^2}{R^2+1)^2} \le \frac{\pi}{R} \to 0 \text{ as } R \to \infty.$$

The function f has poles of order 2 at i and $-i$. Only i is in the upper half plane (where the γ_R are) so we are only interested in the residue at i. Let $g(z) = \frac{z^2}{(z+i)^2}$, so that $f(z) = (z-i)^2 g(z)$. To find the residue of f at i, we calculate $g'(z)$. We get

$$g'(z) = \frac{2z^2 i - 2z}{(z+i)^4},$$

thus we have

$$\text{Res}(f, i) = g'(i) = \frac{1}{2i}.$$

We can now calculate the integral:

$$\int_0^\infty \frac{x}{(x^2+1)^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x}{(x^2+1)^2} dx = \pi i \text{Res}(f, i) = \frac{\pi}{2}.$$

(b) Let $I = \int_0^\infty \frac{x^{1/2}}{1+x^2} dx$ and let $f(z) = \frac{z^{1/2}}{1+z^2}$ where we use the branch of $z^{1/2}$ on $\mathbb{C} \backslash \mathbb{R}_+$ given by $(re^{it})^{1/2} = r^{1/2} e^{it/2}$. We will integrate f over the curve $\gamma_R = [1/R, R] + \{Re^{it} : 0 \le t \le 2\pi\} - [1/R, R] - \{e^{it}/R : 0 \le t \le 2\pi\}$.
Note that on the segment $-[1/R, R]$, $f(x)$ takes the values $\frac{x^{1/2} e^{\pi i}}{1+x^2}$, so on this segment the integral (as $R \to \infty$) is $-e^{\pi i} I = -I$.
The function $f(z)$ has simple poles at $\pm i$. $\text{Res}(f, i) = e^{\pi i/4}/2i$ and $\text{Res}(f, -i) = -e^{-\pi i/4}/2i$.
The integral over the circle $\{Re^{it} : 0 \le t \le 2\pi\}$ is bounded by $2\pi R(R^{1/2})/(R^2+1) < 4\pi R^{-1/2} \to 0$ as $R \to \infty$.
The integral over the circle $\{e^{it}/R : 0 \le t \le 2\pi\}$ is bounded by $2\pi R^{-1}(R^{-1/2})/(R^{-2}+1) < 4\pi R^{-1/2} \to 0$ as $R \to \infty$.
Putting all of this together we get

$$(1 - (-1))I = 2\pi i \left(\frac{e^{\pi/4}}{2i} - \frac{e^{-\pi/4}}{2i} \right)$$

so

$$I = \pi \left(\frac{e^{\pi/4}}{2} - \frac{e^{-\pi/4}}{2} \right) = \pi \cos(\pi/4) = \frac{\pi\sqrt{2}}{2}.$$

(c) Let $I = \int_0^\infty \frac{\log x}{(1+x^2)^2} dx$. Let γ_R be as in (b) and let $f(z) = \frac{\log^2(z)}{(1+z^2)^2} = \frac{\log^2(z)}{(z-i)^2(z+i)^2}$ using the principal branch of log on $\mathbb{C} \backslash \mathbb{R}_+$. The function f has poles of order 2 at $\pm i$ with residues:

$$\text{Res}(f, i) = -\frac{\pi}{4} + \frac{\pi^2 i}{16}$$

and

$$\text{Res}(f, -i) = \frac{3\pi}{4} - \frac{9\pi^2 i}{16}.$$

The integral over the circle $\{Re^{it} : 0 \le t \le 2\pi\}$ is bounded by (for large R)

$$2\pi R(\log R + 2\pi)^2/(R^2 - 1) < 8R^{-1} \log^2 R \to 0 \text{ as } R \to \infty.$$

A similar bound for the integral around $\{e^{it}/R : 0 \le t \le 2\pi\}$, gives that the integral tends to 0 as $R \to \infty$.

Looking at the integral on $[1/R, R] - [1/R, R]$ we get

$$\int_{1/R}^{R} \frac{\log^2 x}{(1+x^2)^2} dx - \int_{1/R}^{R} \frac{(\log x + 2\pi i)^2}{(1+x^2)^2} dx = \int_{1/R}^{R} \frac{4\pi^2 - 4\pi i \log x}{(1+x^2)^2} dx.$$

Thus in the limit we get

$$4\pi^2 \int_0^\infty \frac{dx}{(1+x^2)^2} - 4\pi i I = 2\pi i(-\frac{\pi}{4} + \frac{\pi^2 i}{16} + \frac{3\pi}{4} - \frac{9\pi^2 i}{16}) = \pi^3 + \pi^2 i.$$

Hence $I = -\pi/4$ (by equating the real and imaginary parts of the above equation).

(d) Let $\gamma(x) = e^{ix}$ for $0 \leq x \leq 2\pi$ and set $z = e^{ix}$ so that dz/idx.

$$\begin{aligned}\int_0^{\pi/2} \frac{dx}{a + \sin^2 x} = \frac{1}{4}\int_0^{2\pi} \frac{dx}{a+\sin^2 x} &= \frac{1}{4}\oint_\gamma \frac{1}{a + \left(\frac{z^2-1}{2iz}\right)^2}\frac{dz}{iz} \\ &= i\oint_\gamma \frac{zdz}{z^4 - 2(2a+1)z^2 + 1} \\ &= -2\pi \sum_{|w|<1} \text{Res}\left(\frac{z}{z^4 - 2(2a+1)z^2+1}, w\right).\end{aligned}$$

Solving $z^4 - 2(2a+1)z^2 + 1$ we get $z^2 = 2a + 1 \pm 2\sqrt{a+a^2}$. Since $a > 0$, $0 \leq |2a+1-2\sqrt{a+a^2}| < 1$. The value $2a+1-2\sqrt{a+a^2}$ has two square roots, call them $\pm\alpha$. These are simple roots so

$$\begin{aligned}\text{Res}\left(\frac{z}{z^4 - 2(2a+1)z^2+1}, \pm\alpha\right) &= \frac{z}{4z^3 - 4(2a+1)z}|_{z=\pm\alpha} \\ &= \frac{1}{4\alpha^3 - 4(2a+1)} = \frac{-1}{8\sqrt{a+a^2}}.\end{aligned}$$

Therefore there are two equal residues at $\pm\alpha$. Thus

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = -2\pi \frac{-2}{8\sqrt{a+a^2}} = \frac{\pi}{2\sqrt{a+a^2}}.$$

2. Suppose f is an entire function with $f(\mathbb{C})$ not dense in $\mathbb{C}$. Then there is a $\varepsilon > 0$. and a $w \in \mathbb{C}$ such that, for any $z \in \mathbb{C}$, $|f(z) - w| \geq \varepsilon$. Hence we have

$$0 \leq \left|\frac{1}{f(z) - w}\right| \leq \frac{1}{\varepsilon},$$

i.e. $\frac{1}{f(z)-w}$ is a bounded entire function. It follows by Liouville's Theorem that there is a constant $c \in \mathbb{C}$ such that

$$\frac{1}{f(z) - w} = c.$$

Hence $f(z) = 1/c + w$ is a constant function. Hence if f is a *non-constant*, entire function it must have dense range.

3. Take any $a > 0, a \notin \mathbb{Z}$ and let $f(z) = \frac{1}{z^2+a^2}$. It is clear that f has poles at $z = ia$ and $z = -ia$ and no other singularities. Now, we know from class that

$$\sum_{k=-\infty}^{\infty} f(k) = -\pi(\text{Res}[f(z)\cot(\pi z), ia] + \text{Res}[f(z)\cot(\pi z), -ia]).$$

Hence we get

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} = -\frac{\pi\cot(ia\pi)}{2ia} - \frac{\pi\cot(-ia\pi)}{-2ia} = \frac{\pi}{a}\coth(\pi a),$$

using the fact that $i\cot(iz) = \coth(z)$ for the last equality. Rewriting we get

$$\frac{1}{a^2} + 2\sum_{k=1}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi}{a}\coth(\pi a),$$

and thus

$$\sum_{k=1}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}$$

or equivalently

$$\sum_{k=0}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi}{2a}\coth(\pi a) + \frac{1}{2a^2}.$$

4. Let $I = \int_0^\infty \frac{dx}{1+x^n}$ and let $f(z) = \frac{1}{1+z^n}$. We will integrate f around the curve suggested in the question. The integral over the arc, i.e. over the curve $\{Re^{it} : 0 \le t \le 2\pi i/n\}$, is bounded by $\frac{2\pi R}{n(R^n-1)}$. This tends to 0 as $R \to \infty$.

   The integral over the line segment from $Re^{2\pi i/n}$ to 0 is

$$-\int_0^R \frac{1}{1+(e^{2\pi i/n}x)^n} e^{2\pi i/n}dx = -\int_0^R \frac{1}{1+x^n} e^{2\pi i/n}dx.$$

Thus, in the limit as $R \to \infty$, we get $-e^{2\pi i/n}I$.

   Now f has a simple pole at $e^{\pi i/n}$ and $\mathrm{Res}(f, e^{\pi i/n} = \frac{-e^{\pi i/n}}{n}$. Thus

$$(1 - e^{2\pi i/n})I = \frac{-2\pi i e^{\pi i/n}}{n}$$

and hence

$$I = \frac{2\pi i}{n(e^{\pi i/n} - e^{-\pi i/n})} = \frac{\pi}{n \sin \pi/n}.$$

5. (a) Note that $f(z) = -\frac{1}{z} + \frac{1}{z^2-1} = -\frac{1}{z} + \frac{1}{2(z+1)} + \frac{1}{2(z-1)}$.

   On $\mathbb{A}_1$ we have

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n,$$

$$\frac{1}{2(z-1)} = 2(z-1)^{-1}$$

   and

$$\frac{1}{2(z+1)} = \frac{1}{4}\cdot\frac{1}{1+\frac{z-1}{2}} = \sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^{n+2}(z-1)^n.$$

   Thus, the Laurent series for f is

$$f(z) = 2(z-1)^{-1} + \sum_{n=0}^{\infty}[(-1)^{n+1} + (-1)^n/2^{n+2}](z-1)^n.$$

   On $\mathbb{A}_2$ we have

$$\frac{1}{z} = \frac{-1}{1-z}\cdot\frac{1}{1-\frac{1}{1-z}} = -\sum_{n=-1}^{-\infty}(1-z)^n,$$

$$\frac{1}{2(z-1)} = 2(z-1)^{-1}$$

   and

$$\frac{1}{z+1} = \frac{1}{z-1}\cdot\frac{1}{1+\frac{2}{z-1}} = \sum_{n=-1}^{-\infty}(-1)^{n-1}2^{-n-1}(z-1)^n.$$

Thus, the Laurent series for f is

$$f(z) = (2 + 2^{-2} - 1)(z-1)^{-1} + \sum_{n=-2}^{-\infty} [(-1)^{n-1}2^{-n-1} - 1](z-1)^n.$$

On $\mathbb{A}_3$ we have

$$\frac{1}{z+1} = \sum_{n=0}^{\infty} (-1)^n z^n$$

and

$$\frac{1}{z-1} = -\sum_{n=0}^{\infty} z^n.$$

Thus, the Laurent series for f is

$$f(z) = -z^{-1} - \sum_{n=0}^{\infty} z^{2n+1} = -\sum_{n=-1}^{\infty} z^{2n+1}.$$

(b) This curve is just a stretched circle (an ellipse). It winds around 0 and 1 four times. It goes around -1 zero times.

(c) To find this integral we need to find the residues at 0 and 1 (we don't need the residue at -1 since this is outside the curve). Calculate the $\mathrm{Res}(f, 0) = -1$ and $\mathrm{Res}(f, 1) = 2$. Hence

$$\int_\gamma f(z)dz = 2\pi i(4 \times -1 + 4 \times 2) = 8\pi i.$$

6. (a) $\cot(z) = \frac{\cos z}{\sin z}$. The singularities of $\cot z$ are at the zeroes of $\sin z$. Now $\sin z = 0$ if and only if $e^{iz} - e^{-iz} = 0$ if and only if $e^{2iz} = 1$. Hence $\sin z = 0$ if and only if $z = n\pi$ for an integer n. The derivative of sin is cos and for any integer n $\cos n\pi \neq 0$. Hence these are all simple zeroes for sin. Hence $n\pi$ is a pole of order 1 for $\cot z$.

(b) The function $\sin z$ is entire, so $z \sin(1/z)$ is analytic on $\mathbb{C}\backslash\{0\}$. Using the expansion of $\sin z$ about 0 we get

$$z \sin(1/z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-2n}$$

Since there are an infinitely many non-zero terms, $z \sin(1/z)$ has an essential singularity at 0.

(c) $f(z)$ is analytic on $\mathbb{P}\backslash\{1\}$. The Taylor expansion of $\log z$ about 1 is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$$

with radius of convergence 1. Hence the Laurent expansion of f about 1 is

$$f(z) = \sum_{n=-3}^{\infty} \frac{(-1)^{n-1}}{n+4}(z-1)^n$$

hence there is a pole of order 3 at 1.